# Spontanous decay

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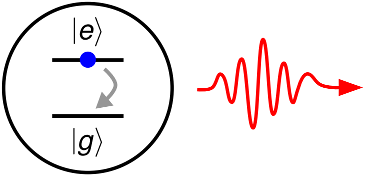
# Outline

# Spontanous decay in free space




We consider a two level atom with ground and excited states respectively $|g\rangle$ and $|e\rangle$.

We will calculate the rate at which an atom spontaneously relaxes to a lower energy state while emitting a photon.

# Initial state

$$|\psi(0)\rangle = |e, \{0\}\rangle$$

- the atom starts in excited state $|e\rangle$
- the radiation field consists of a continuum of modes which we think of as a cavity which is infinite in extend
- the radiation field starts in vacuum state $|\{0\}\rangle$

# Final state

$$|\psi(\infty)\rangle = \left|g, 1_{\vec{k}}\right\rangle$$

The atom will eventually decays to the ground state $|g\rangle$ emitting a photon in mode $\vec{k}$.

# Time dependent state

$$|\psi(t)\rangle = C_0^e(t)e^{-i\omega_0 t}\,|e,\{0\}\rangle + \sum_{\vec{k}} C^g_{1\vec{k}}(t)e^{-i\omega_k t}\,\left|g, 1_{\vec{k}}\right\rangle$$

Linear combination of the initial and final states provides a complete basis to describe the time-dependent state of the system.

# Outline

# The Hamiltonian of the system consists of three parts

$$\hat{H} = \underbrace{\hbar\omega_0 \left|e\right\rangle\!\left\langle e\right|}_{H_A} + \underbrace{\sum_{\vec{k}} \hbar\omega_k \hat{a}^{\dagger}_{\vec{k}} \hat{a}_{\vec{k}}}_{H_F} + \underbrace{\sum_{\vec{k}} \hbar g_{\vec{k}} \left( \left|e\right\rangle\!\left\langle g\right| \hat{a}_{\vec{k}} + \left|g\right\rangle\!\left\langle e\right| \hat{a}^{\dagger}_{\vec{k}} \right)}_{H_I}$$

with atom-field coupling:

$$g_{\vec{k}} = i\sqrt{\frac{\omega_k}{2\hbar\varepsilon_0 V}}\left(\vec{d}\cdot\vec{\epsilon}_{\vec{k}}\right)$$

E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963)

# Time evolution of the system: Schrödinger equation

$$\hat{H}\,|\psi(t)\rangle = i\hbar\frac{\partial\,|\psi(t)\rangle}{\partial t}$$

Projecting onto $\langle e, \{0\}|$ and $\langle g, 1_{\vec{\mathbf{k}}}|$

$$\frac{\partial}{\partial t}C_0^e(t) = i\sum_{\vec{\mathbf{k}}} g_{\vec{\mathbf{k}}}e^{-i(\omega_k-\omega_0)t}C^g_{1\vec{\mathbf{k}}}(t) \tag{1}$$

$$\frac{\partial}{\partial t}C^g_{1\vec{\mathbf{k}}}(t) = ig^*_{\vec{\mathbf{k}}}e^{i(\omega_k-\omega_0)t}C_0^e(t) \tag{2}$$

# Integration time!

We first formally integrate (2)

$$C^{g}_{1\vec{k}}(t) = i g^{*}_{\vec{k}} \int_0^t \mathrm{d}t' \, e^{i(\omega_k - \omega_0)t'} C^{e}_{0}(t')$$

And then replace into (1)

$$\frac{\partial}{\partial t} C^{e}_{0}(t) = -\sum_{\vec{k}} \left| g_{\vec{k}} \right|^2 \int_0^t \mathrm{d}t' \, e^{i(\omega_k - \omega_0)(t-t')} C^{e}_{0}(t')$$

# From discrete to continuum

For $V \to \infty$ the discrete set of modes tends to a continuum

$$\sum_{\vec{\mathbf{k}}} \to \overbrace{2}^{\text{2 pol per mode}} \int \mathrm{d}^3k \underbrace{D(k)}_{\frac{V}{(2\pi)^3}}$$

$$\sum_{\vec{\mathbf{k}}} \to \frac{2V}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{\infty} k^2 \, \mathrm{d}k$$

# Let's get rid of $\sum_{\vec{k}} \left|g_{\vec{k}}\right|^2$

$$\sum_{\vec{k}} \left|g_{\vec{k}}\right|^2 \to \int_0^\infty \mathrm{d}k\, k^2 \frac{\omega_k}{(2\pi)^3 \hbar \varepsilon_0} \underbrace{\left[\int_0^\pi \sin\theta\, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \left(\vec{\mathbf{d}} \cdot \vec{\epsilon}_{\vec{k}}\right)^2\right]}_{\frac{8\pi}{3}\left|\vec{\mathbf{d}}\right|^2}$$

using $\omega_k = c\, k$:

$$\sum_{\vec{k}} \left|g_{\vec{k}}\right|^2 \to \frac{\left|\vec{\mathbf{d}}\right|^2}{6\pi^2 \hbar \varepsilon_0 c^3} \int_0^\infty \omega_k^3\, \mathrm{d}\omega_k$$

Let's go back to the time integral

$$\int_0^t \mathrm{d}t'\, e^{i(\omega_k - \omega_0)(t-t')} C_0^e(t')$$

# Wigner-Weisskopf approximation

We assume that $C_0^e(t)$ varies in a timescale $\Gamma \ll \omega_0$

$$C_0^e(t') \to C_0^e(t)$$

And we can take it out of the integral

$$C_0^e(t) \int_0^t \mathrm{d}t' \, e^{i(\omega_k - \omega_0)(t-t')}$$

This is a Markov approximation:
the system has no memory of the past.

V. Weisskopf and E. Wigner, Zeitschr. Phys. **63**, 54 (1930)

# Time integration

We're interested in time scales $t \gg 1/\omega_0$, then $\int_0^t \to \int_0^\infty$

$$\int_0^\infty \mathrm{d}t'\, e^{i(\omega_k-\omega_0)(t-t')} = \pi\delta(\omega_k-\omega) - iP\left(\frac{1}{\omega_k-\omega}\right)$$

where $P$ is Cauchy principal part.

Complex values correspond to frequency shifts (Lamb shift).
It's nasty, it diverges. Let's forget about it...

# Combining temporal and momentum part

$$\frac{\partial}{\partial t} C_0^e(t) = -\underbrace{\sum_{\vec{\mathrm{k}}} \left| g_{\vec{\mathrm{k}}} \right|^2}\ \underbrace{\int_0^t \mathrm{d}t'\, e^{i(\omega_k - \omega_0)(t-t')} C_0^e(t')}$$

$$\frac{\partial}{\partial t} C_0^e(t) = \overbrace{\frac{\left|\vec{\mathrm{d}}\right|^2}{6\pi^2 \hbar \varepsilon_0 c^3} \int_0^\infty \mathrm{d}\omega_k\, \omega_k^3}\ \overbrace{\pi \delta(\omega_k - \omega_0)\, C_0^e(t)}$$

# Excited state amplitude time evolution

$$\frac{\partial C_0^e(t)}{\partial t} = -\frac{\Gamma}{2} C_0^e(0) \quad \text{with} \quad \Gamma = \frac{\left|\vec{\mathbf{d}}\right|^2 \omega_0^3}{3\pi\hbar\varepsilon_0 c^3}$$

this one with can solve:

$$C_0^e(t) = e^{-\frac{\Gamma}{2}t} C_0^e(0) \tag{3}$$

The excited state amplitude decays exponentially with rate $\Gamma$, AKA Einstein *A* coefficient

# Spectrum of the emitted light

We can now solve Eq. (2)

$$\frac{\partial}{\partial t} C^{g}_{1\vec{\mathbf{k}}}(t) = i g^{*}_{\vec{\mathbf{k}}} e^{i(\omega_k - \omega_0)t} e^{-\frac{\Gamma}{2}t} C^{e}_{0}(0)$$

The process takes an infinite time to complete. Integrating:

$$\lim_{t\to\infty} \left| C^{g}_{1\vec{\mathbf{k}}}(t) \right| = \frac{\left| i g_{\vec{\mathbf{k}}} \right|}{\frac{\Gamma^2}{4} + (\omega_k - \omega_0)^2}$$

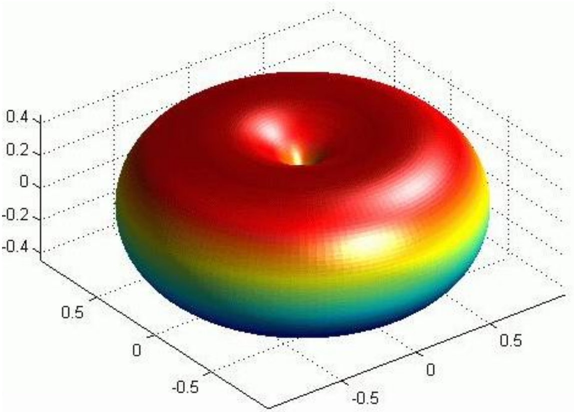
We find a Lorentzian spectrum.

What a surprise, eh?

# Spatial distribution

The $\left|ig_{\vec{k}}\right|$ term contains the spatial information. For $\Delta m = 0$

$$\lim_{t\to\infty}\left|C^{g}_{1\vec{k}}(t)\right| \sim \sum_{s}\left|\vec{d}\cdot\vec{\epsilon}\right|^{2} \propto \sin^{2}(\theta)$$



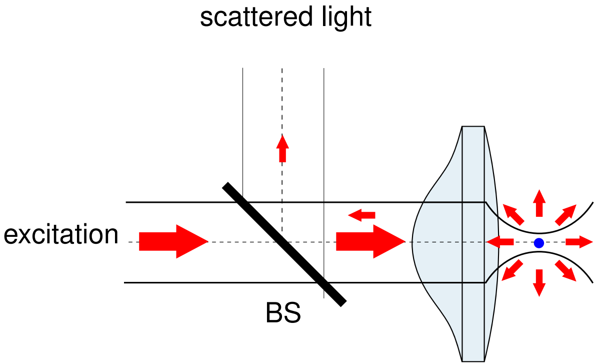
# Outline

The problem

Wigner-Weisskopf Theory

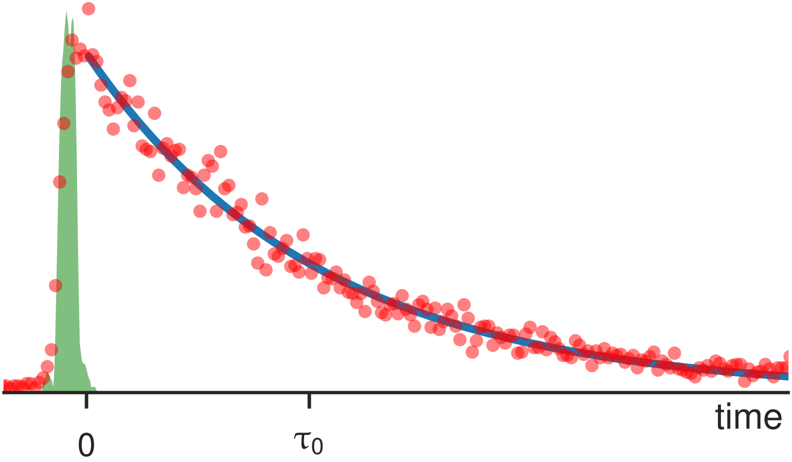
A simple experiment

# Pulse excitation of a single atom




B. Darqui, *et al.*, Science **309**, 454 (2005)

# Spontanous decay of $^{87}$Rb




V. Leong, *et al.*, Phys. Rev. A **91**, 063829 (2015)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