Fibre Sensing

Tay Wei Yau, Thormund
A0136214R
Department of Physics, Faculty of Science
National University of Singapore

Supervisor:
Professor Christian Kurtsiefer

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Abstract

A fiber-based Michelson interferometer is capable of measuring sub-micrometer changes in its optical path difference, and can be utilized for sensing mechanical perturbations in the environment. Such an interferometer can be modified to a heterodyned phasemeter by placing an acousto-optical frequency shifter in one of the interfering arms, which allows unambiguous measurements of the directional displacements in the optical path difference.

This thesis describes the development and optimizations of such a fiber-based heterodyne phasemeter operating at telecommunication wavelength (1550 nm). In this phasemeter, the optical path differences between the two interfering arms are encoded into the phase of a beatnote between two different optical frequencies. The phase information is extracted through a digital I/Q demodulation scheme, which involves sampling the beat signal at a specific frequency. This allows for an accurate reading of the length difference between the two arms of the interferometer.
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Chapter 1

Introduction

1.1 Recent Developments in Underwater Seismology

71% of the Earth’s surface is covered by water, posing major logistical challenges for geophysical research in the Earth Science community. As such, majority of seismic stations are located on land, with limited numbers of temporary ocean-bottom seismometers\cite{1, 2}, and permanently tethered observatories\cite{3}. However, these systems are very costly and often unfeasible on large scales.\cite{4}

With the rise of the internet, most oceans are crossed by an extensive infrastructure of telecommunication fibres.\cite{5} In 2012, an international Joint Task Force was established to design “SMART” (Scientific Monitoring and Reliable Telecommunications) cables with environmental sensors embedded in repeaters every $\sim 50$ km.\cite{6, 7} Since, there has been efforts to turn these optical fibres into seismo-acoustic sensors. A breakthrough is shown in Marra et al. \cite{4}, coincidental detection of earthquakes via analyzing the phase stability of the metrology-grade lasers has shown this method to be successful with optical fibres with lengths on the order of 1000 km. However, the author acknowledges that in the case of a single optical fibre, there is ambiguity in determining the location of seismic events, which can only be resolved with the use of a second optical fibre located elsewhere.

An alternative method known as Distributed Acoustic Sensing (DAS)\cite{8–11}, analyses backscattered laser light with phase-based optical time domain-reflectometry\cite{12} to resolve seismic activity detected to within the fibre\cite{8, 9}. This method has been successful at mapping out previously unmapped fault zones with a single optical fibre.\cite{8} However, DAS as a method is only feasible with much shorter optical cables, on the order of 20 km due to attenuation.\cite{12, 14}

With these considerations, this thesis proposes the development and improvement of a fibre-based interferometric system that can detect mechanical perturbations.
1.2 Thesis Outline

The thesis aims to describe the development and optimisations of a digital fibre-based interferometric system, called the heterodyne phasemeter [15].

In Chapter 2, the well-known Michelson interferometer [15] is first introduced, before highlighting on its limitations. This gives rise to the modification of the interferometer to produce a heterodyned phasemeter in Chapter 3. However, the signal from heterodyned phasemeter is modulated unlike the Michelson interferometer, and a demodulation scheme—I/Q demodulation is introduced in theory.

Chapter 4 discusses some experimental results that follows from using I/Q demodulation, and complications that arises in such an experimental setup. Chapter 5 then experiments with a different demodulation scheme, and describes some of the results.
Chapter 2

Michelson Interferometer Setup

2.1 Fibre-based Michelson Interferometer

Figure 2.1 is a schematic diagram of a fibre-based Michelson interferometer. We begin with the use of a coherent light source—a laser. Denote the electric field into the interferometer by $E_{\text{in}}$, with power $P_{\text{in}}$, with a wavelength $\lambda_0 = 1550$ nm. The optical index of the optical fibre core has value $n = 1.52$, and the wavenumber of the light within the circuit is $k = nk_0 = n\frac{2\pi}{\lambda_0}$.

To avoid back reflection into the laser, the light is passed through an optical isolator. The laser light is then passed through a 50:50 coupler. Here, the coupler acts as a beam splitter, and the laser light passes through both fibre-arms of the interferometer, with lengths $L_x$ and $L_y = L_x + \Delta L$.

Due to birefringence [16][17], the state of polarization is altered as it travels through an optical fibre. If the light was simply reflected off a regular mirror, the light that exits each arm of the Michelson interferometer could be in sufficiently distinct polarization states such that the interference may not produce a distinct interference pattern. This can be compensated with the use of a Faraday mirror at the end of each arm of the Michelson interferometer. In a Faraday Mirror, the electric field gains a $\pi/4$ rad rotation in polarization state as it passes between the Bismuth Iron Garnet (BIG) Faraday Rotating element, before being reflected off the mirror, and another $\pi/4$ rad as it passes through the BIG magnets once more. As such, light that reflects off a Faraday mirror has an effective $\pi/2$ rad rotation in its state of polarization with respect to the input state. Any perturbation that is picked up by the laser light from the birefringence in the optical fibre will therefore be reversed during the return trip. As such, the laser light
that is emitted from both arms of the interferometer has polarization state that is orthogonal to the light that had entered each arm. This allows for a strong interference between the laser light from both arms of the interferometer.

Here, the light would have traveled down $2L_x$ in one arm, and $2L_x + 2\Delta L_x$ in the other arm. They superpose together at the 50:50 coupler, and exits the interferometer to be detected by the photodiode. The electric field at the output of the interferometer is given by

$$E_{\text{out}} = \frac{1}{2} E_{\text{in}} \left( e^{i2kL_x} + e^{i2kL_y} \right).$$

At this point, it is convenient to define the optical phase difference

$$\phi_d = 2k\Delta L,$$  \hspace{1cm} (2.1)

which allows us to rewrite the outgoing electric field and measurable power as

$$E_{\text{out}} = \frac{1}{2} E_{\text{in}} e^{i2kL_x} \left( 1 + e^{i\phi_d} \right),$$  \hspace{1cm} (2.2)

$$P_{\text{out}} = \frac{P_{\text{in}}}{4} \left[ 4 \cos^2 \left( \frac{\phi_d}{2} \right) \right] = \frac{P_{\text{in}}}{2} \left[ 1 + \cos \phi_d \right].$$  \hspace{1cm} (2.3)

The output optical power from the interferometer is therefore proportional to the laser power into the interferometer, and is a sinusoidal of the optical phase difference $\phi_d$.

**Ambiguous Relative Length Changes** A direct observation of Equation (2.3) shows that the power measured by the photodiode is periodic with respect to the optical phase difference $\phi_d$, and consequentially, periodic with respect to the relative length difference between the 2 arms of the interferometer. A graphical interpretation is as given in Figure 2.2.

![Figure 2.2: Visual representation of power against optical phase difference](image)

If the power fluctuation due to the length changes is small, there is no issue with such an experimental setup to detect relative length changes, provided the initial outgoing power is far from the extrema points as circled within Figure 2.2.

However, at the extrema of the cosine function, such as those circled within Figure 2.2 any change (both increase and decrease) in $\Delta L$, produces the same change in $P_{\text{out}}$. As such, there an is ambiguity in the increase or decrease in $\Delta L$ for a given change in $P_{\text{out}}$ from its extrema. With extended measurement, there is greater ambiguity in the relative length changes $\Delta L$ of the fibre if the output optical power from the interferometer is near an extrema point.
Chapter 3

Heterodyned Phasemeter and Demodulation

This section will first describe my experimental setup to modulate the optical phase difference into a carrier signal, followed by 2 different methods to demodulate the phase information from the carrier signal.

3.1 Heterodyne Phasemeter Setup

While a conventional Fibre-based Michelson interferometer is able to measure sub-micrometer changes in optical fibre length, it is ambiguous at measuring the displacement near the extrema in output power. The heterodyned phasemeter is able to solve this issue. An unambiguous optical phase $\phi_d$ (equivalently, displacement $\Delta L$) is made possible by interfering electric fields with two different frequencies within the interferometer. A different optical frequency from the input laser can be provided through a frequency shifter, based on an acousto-optic modulator (AOM). Figure 3.1 shows a schematic of my experimental setup.

In contrast to the fibre-based interferometer, this experimental setup includes the use of an AOM in the control arm of the ‘interferometer’, giving a phasemeter. This phasemeter is said to be ‘heterodyne’, as it utilises 2 frequencies to determine the phase of the interference signal.

Similar to the previous chapter, the laser light entering the interferometer is first split into
CHAPTER 3. HETERODYNED PHASEMETER AND DEMODULATION

both arms of the interferometer by the 50:50 coupler. Along the control arm, the AOM is connected to a frequency reference, which is set to some frequency $f_{AOM}$. The light in the control arm, before passing through the AOM, has some angular frequency $\omega = kc$. The first order diffraction through the AOM gains the angular frequency $\Omega = 2\pi f_{AOM}$ to then have a total angular frequency of $\omega + \Omega$, before traveling further down the optical fibre and reflected off the Faraday mirror. The same process is repeated once more as it passes down the AOM a second time on its way to the 50:50 coupler, giving rise to a total angular frequency of $\omega + 2\Omega$ in the light outgoing from the control arm of the interferometer.

The light from the control arm then interferes with light from the sensing arm at the 50:50 coupler, which creates an optical beat signal.

**Expected Signal** By considering the linear superposition of the outgoing light from each interferometer arm, the expected electric field as a function of time is given by

$$E_{\text{out}}(t) = \frac{1}{2} E_{\text{in}} \left( e^{i(\omega t - \phi_d/2)} + e^{i(\omega t + 2\Omega t + \phi_d/2)} \right).$$

Utilising the optical phase difference definition in Equation (2.1), the equation is simplified to

$$E_{\text{out}}(t) = \frac{1}{2} E_{\text{in}} e^{i(\omega t - \phi_d/2)} \left( 1 + e^{i(2\Omega t + \phi_d)} \right).$$

Therefore, the power $P_{\text{out}}$ that exits the phasemeter is given by

$$P_{\text{out}}(t) = \frac{P_{\text{in}}}{2} [1 + \cos (2\Omega t + \phi_d)].$$

As such, for fixed optical phase difference $\phi_d$, the output power from the phasemeter is an optical beat signal with frequency $f = 2\Omega/2\pi$.

**Physical Devices Used** The laser source used in this project is a Thorlabs SFL1550S laser. This laser is based on an external cavity configuration, with a lasing wavelength of 1550 nm. In the given configuration, the laser generates a power output approximately 5 mW. The power that exits the interferometer is approximately 2 mW. The laser linewidth was measured utilising the self-heterodyne method. This method involves optically beating the laser with a delayed portion of itself in an incoherent manner. The provided delay needs to be larger than the estimated coherence time of the laser itself.

In this case, we set the interferometer to an imbalanced configuration, with one arm connected to 1 km long fibre spool, and the other arm to a 6 m fibre patch-cord. The power spectrum of the optical beatnote was measured with a spectrum analyzer, and is shown in Figure 3.2.

When the light interferes with itself incoherently, the measured linewidth of the optical beatnote is twice of the natural linewidth of the laser itself. In Figure 3.2, the spectrum was fitted with a lorentzian profile, and the full width at half maximum (FWHM) was found to be about 490 kHz. As such, we deduce that the laser linewidth is about 245.1(9.2) kHz, which corresponds to a coherence length of about 389 m.

The AOM used in this setup is from Brimrose (AMF-80-20-1550). At different driving frequencies, the first order diffraction efficiency of the AOM is shown in Figure 3.3.
3.2. I/Q DEMODULATION BY ANALOGUE CIRCUIT

In the previous section, it has been shown that phase information (optical phase difference $\phi_d$) has been encoded as the phase of the optical beatnote. In this section, a scheme that demodulates this optical beat signal to obtain the phase information will be described.

This demodulation process is commonly known as I/Q demodulation, and an I/Q demodulator can be constructed using analogue components, such as mixers and low pass filters. An
example of such a demodulator is shown in Figure 3.4. A direct digital synthesizer (DDS) generates a strong sinusoidal signal at twice of the AOM driving frequency ($f = 2f_{\text{AOM}} \approx 140$ MHz). This signal, also known as the local oscillator, is split into two via a power splitter. The two split signals are directed into two RF mixers, with one signal being phase delayed by $\pi/2$ rad relative to the other signal. In the meantime, the optical beat signal measured from the photodiode is also split and sent into the two mixers. The mixer performs a multiplication between the two inputs.

For the first mixer (cosine mixer in Figure 3.4), the output signal is given by

$$P_{\text{out}}(t) \cdot \cos 2\Omega t = \frac{P_{\text{in}}}{2} [2 \cos 2\Omega t + \cos(4\Omega t + \phi_d) + \cos \phi_d],$$

where $2\Omega = 2\pi f = 4\pi f_{\text{AOM}}$. In the other mixer (sine mixer in Figure 3.4), the output signal is given by

$$P_{\text{out}}(t) \cdot \sin 2\Omega t = \frac{P_{\text{in}}}{2} [2 \sin 2\Omega t + \sin(4\Omega t + \phi_d) - \sin \phi_d].$$

We can easily observe that the output signal in the previous expressions from both mixers contain not only the sine/cosine value of the optical phase difference $\phi_d$, but also the local oscillator frequency $2\Omega$ and its higher harmonic. The high frequency terms ($2\Omega, 4\Omega$) can be easily removed with an appropriate low pass filter, leaving only the $\cos \phi_d, -\sin \phi_d$ terms. This gives the $I, Q$ components to be

$$I = \frac{P_{\text{in}}}{2} \cos \phi_d, \quad Q = -\frac{P_{\text{in}}}{2} \sin \phi_d, \quad (3.3)$$

and optical phase difference calculated to be

$$\phi_d = \tan^{-1}\left(\frac{-Q}{I}\right). \quad (3.4)$$
From the previous section, note that measuring the phase allows one to determine the length changes within the interferometer.

![Figure 3.5: A sinusoidal wave can be represented by a rotating vector of amplitude $A$, with phase $\phi$.](image)

With reference to Figure 3.5, a sinusoidal wave can be represented by a rotating vector, with its amplitude being the length of the vector $A$, and its phase being the argument of the vector $\phi$. The $I, Q$ components are merely projections of this vector onto the $x, y$ axes. This representation is commonly known as a phasor diagram.

### 3.3 Digital I/Q Demodulation

While such a circuit as described in Section 3.2 is capable of demodulating the signal, the setup requires an assembly of passive components and a physical RF source as the local oscillator. While this physical setup can be made compact, there exists a digital alternative which significantly simplifies the demodulation procedure. For this section, a general observation is made on sinusoidal functions, before explaining its implication on our experimental setup, and hence, the choice of sampling scheme.

**Mathematical Description**  We begin with a different interpretation of the I/Q components of a signal. Assume we have an arbitrary sinusoidal signal $y(t)$ with an angular frequency $\omega = 2\pi f$, and an amplitude $A$. One can express $y(t)$ as

\[
y(t) = A \sin(\omega t + \varphi_0) \\
= A \cos \varphi_0 \sin \omega t + A \sin \varphi_0 \cos \omega t, \\
= I \sin \omega t + Q \sin(\omega t - \pi/2).
\]

Here, we see that the quadrature-phase term $Q$ lags the in-phase term $I$ by $\pi/2$. 
If one measures this signal $y(t)$ at times $t$ such that $\omega t$ is a multiple of $\pi/2$, one can easily extract the $I, Q$ components, as exemplified in Figure 3.6, where\[19\]

\begin{align*}
\omega t_0 &= 0, \quad y(t_0) = I; \\
\omega t_1 &= \pi/2, \quad y(t_1) = Q; \\
\omega t_2 &= \pi, \quad y(t_2) = -I; \\
\omega t_3 &= 3\pi/2, \quad y(t_3) = -Q.
\end{align*}

There is an implicit assumption that the amplitude $A$ and phase $\varphi$ of the signal does not change abruptly between consecutive samples.

With some observation of Figure 3.6 (or perhaps the $2\pi$ periodicity of $\omega t \to \omega t + 2\pi$ on Equation (3.5)), one would be able to make the justified remark that the $Q$ value as read off $y(t_1)$ where $0.25 \cdot 2\pi$ to be no different from the $Q$ value as read off $y(t_5)$ where $1.25 \cdot 2\pi$.

Explicitly, for any sampled value at time $t_i$, the next value can be sampled at time $t_{i+1}$ with $\omega (t_{i+1} - t_i) = (m + 1/4) \cdot 2\pi$, where $m$ is a natural number. As such, the sampling frequency $f_s$ no longer needs to be 4 times the frequency of the sinusoidal, but any frequency $f_s$ that satisfies

$$f_s = \frac{N}{M} f,$$

with rational $N/M$, such that $N = 4$.\[19\]

Regardless of the time $t_i$ where $\omega t_i$ is some multiple of $\pi/2$ when the sinusoidal was measured, the phase offset $\varphi_0$ is always given by

$$\varphi_0 = \tan^{-1} \left( \frac{Q}{I} \right)$$

in this convention.

**Digital I/Q Demodulation with Lower Sampling Frequencies** With reference to the mathematical description, the experimental setup demands that the output optical beat needs to have a stable amplitude, and the change in optical path difference in the interferometer has to happen at a much lower frequency than the AOM and sampling frequency.

In our experimental setup, the shift in optical frequency is $f = 2f_{AOM} \approx 140\text{ MHz}$. This
translates to a high sampling frequency $f_s = 4f \approx 560\text{ MHz}$, by constraint of $\Delta t = t_{i+1} - t_i = \pi/2\omega$ in the initial description. Such a high sampling frequency which will generate an excessive amount of data. As such, we wish to reduce the sampling frequency, which the mathematical description permits by choice of $f_s = \frac{4}{M}f$, where $M$ is an integer one is free to choose.

**Sampling Scheme** This gives rise to our choice of sampling scheme. With reference to Equation (3.2), we expect the measurement on the power emitted from the interferometer to be a sinusoidal with frequency $f \approx 140\text{ MHz}$. As prior established, while it is possible to choose a sampling frequency $f_s$ to be $4f \approx 560\text{ MHz}$, at this high frequency the setup generates a dense data set. We can instead choose a more conservative lower sampling frequency, such as on the order of MHz. In the case of sampling frequency $f_s = 1.0\text{ MHz}$, we therefore detune the driving frequency of the AOM to $f_{AOM} = 70.125\text{ MHz}$. Here, $N = 4$, and $M = 561$. Experimental justification for such a choice of sampling frequency is provided in Section 4.1, where the scheme was shown to have been successful.

The sampling can be done by any analogue to digital converter (ADC), at a sampling frequency $f_s = \frac{4}{M}f_{AOM}$ attached to it.

### 3.4 From Phase to Relative Length Changes

At this point, the discussion for obtaining the optical phase difference $\phi_d$ under the scheme of I/Q demodulation is in principle complete. However, the definition of optical phase difference, as given in Equation (2.1), may falsely elude that the problem of determining the relative length changes $\Delta L$ is also in principle well-determined, when it is not. A hint as to why this is the case is also found in the reason to the choice of sampling scheme, where phase is a $2\pi$ periodic quantity. Specifically, Equation (3.6) yields only the principle value. Formally, the arctangent function $\tan^{-1}$ maps the reals $\mathbb{R}$ to the interval $(-\pi, \pi)$. In other words, the optical phase difference $\phi_d$ as provided in

$$\phi_d = 2k\Delta L \quad \in \mathbb{R}, \quad (2.1 \text{ modified})$$

$$\phi_d = \tan^{-1}\left(\frac{Q}{I}\right) \quad \in (-\pi, \pi), \quad (3.6 \text{ modified})$$

are not the same.

Therefore, we have to “unwrap” the measured phase to obtain the relative length changes $\Delta L$, which is certainly a quantity much larger than $\pi/k$.

As prior asserted in Chapter 2, we expect some change in the optical phase difference $\phi_d$ over time. Hence, it is sensible to view the complex phasor as having $I, Q$ coordinate at time $t_i$ as given by $(I_i, Q_i) = (y_i, y_{i+1})$. However, this would mean that we are measuring the absolute phase as the phasor walks along the complex circle, instead of the “phase offset” $\varphi_0$ associated with “$t = 0$”, which slightly differs from the previous section. This view of the $I, Q$ components as coordinates is introduced here to be consistent with Chapter 5.

---

1. This convention is chosen to be in line with ISO/IEC 9899:1999.
When viewing the phasor as having coordinate \((I_i, Q_i)\), by this sampling scheme, we expect a \(\pi/2\) walk along the complex circle of identical radius after each sample. This gives successive absolute phase measurements \(\phi_0, \phi_1, \phi_2, \cdots \in (-\pi, \pi]\) associated with time \(t_0, t_1, t_2, \cdots\), where we expect
\[
\phi_{i+1} - \phi_i \approx \frac{\pi}{2}
\]  
(3.7)
if the measured phase was not constrained to the interval \((-\pi, \pi]\). However, since the measured phase is constrained, we can make an educated guess that
\[
\phi_{i+1} - \left(\phi_i + \frac{\pi}{2}\right) \approx \begin{cases} 
-2\pi & \text{if } \phi_{i+1} \text{ "crosses above" } \pi, \\
0 & \text{if } \phi_{i+1} \text{ does not "cross above" } \pi.
\end{cases}
\]
(3.8)

Pictorially, we express this intuition as the Figure 3.7.

![Figure 3.7: Pictorial demonstration of unwrapping 2\(\pi\) periodicity. Blue dots represent phase if there is no relative length changes, while red dots show the phase obtained from \(\tan^{-1}\).](image)

Accounting for relative length changes \(\Delta L_i\) between time \(t_i\) and time \(t_{i+1}\) that leads to phase difference \(\Delta \phi_i\), one can explicitly write as
\[
(\phi_{i+1} - \Delta \phi_i) - \left(\phi_i + \frac{\pi}{2}\right) = \begin{cases} 
-2\pi & \text{if } \phi_{i+1} \text{ "crosses above" } \pi, \\
0 & \text{if } \phi_{i+1} \text{ does not "cross above" } \pi.
\end{cases}
\]
(3.9)

Rewriting \(\Delta \phi_i\) as the subject, we can say that a relative length change (as measured through \(\Delta \phi_i\)) between time \(t_i\) and \(t_{i+1}\) to be
\[
\Delta \phi_i = \begin{cases} 
\phi_{i+1} - \phi_i - \frac{\pi}{2} & \text{if } |\phi_{i+1} - \phi_i - \frac{\pi}{2}| < |\phi_{i+1} - \phi_i - \frac{\pi}{2} + 2\pi|, \\
\phi_{i+1} - \phi_i - \frac{\pi}{2} + 2\pi & \text{else.}
\end{cases}
\]
(3.9)
Combining everything together, we get the phase differential over time to be

\[ \phi_d(t_{i+1}) = \phi_0 + \sum_{n=0}^{i} \Delta \phi_n, \]

and the issue in measuring relative length changes, as presented with the Fibre-base Michelson Interferometer is in principle nullified.

**Choice of \( f_s \)** Nonetheless, it remains clear that the choice of sampling frequency \( f_s \) still needs to be frequent enough, such that \( \phi_{i+1} - \phi_i \) is reasonably small. By the sampling scheme, it is also clear that the setup cannot distinguish very large length changes such that \( \phi_{i+1} - \phi_i \gtrsim 2\pi \) from small length changes; and the choice of sampling frequency should account for perturbations one would expect.

**Optimisations** Due to the nature of computer code, the use of conditional operators (i.e.: if ... else ...) severely slows down the rate of processing. A derivation by observation is done in Appendix A.1, and Equation (3.9) can instead be written as

\[ \Delta \phi_i = \left( \phi_{i+1} - \phi_i - \frac{\pi}{2} \right) + 2\pi H \left( \pi + \phi_{i+1} - \phi_i - \frac{\pi}{2} \right). \]
Chapter 4

I/Q Demodulation Results

With a scheme laid out in the previous chapter to obtain the optical phase difference $\phi_d$ from the power that exits the phasemeter

$$P_{\text{out}}(t) = \frac{P_0}{2} [1 + \cos (2\Omega t + \phi_d)],$$  (3.2)

we proceed to test if the proposed scheme is able to perform as expected.

4.1 Initial Proof of Concept

An immediately obvious test one can perform, is to ‘knock’ near the phasemeter setup gently. The phasemeter output was connected to a photodiode. The photodiode output signal is then connected to an oscilloscope sampling at 100 kS/s. Meanwhile, the AOM is driven with a frequency $f_{\text{AOM}} = 80.0125$ MHz.

![Figure 4.1: A plot of $\phi_d$ over time, from a period of relative quiet to signal picked up from tapping the table. A distinct ringdown and an envelope can be seen.](image)

Table tap, $f_s = 100.0$ kS/s, $2f_{\text{AOM}} = 160.025$ MHz, $N = 4$

After the oscilloscope was set to record, I tapped the table which the phasemeter setup was on, and processed the data in accordance with the scheme described in the previous chapter. The result is shown in Figure 4.1. The strength of the tap is approximately comparable to typing on a keyboard.
The distinct ringdown and fairly clear resolution of the demodulated signal by the scheme presented in Section 3.3 is an indication that the scheme is successful. Furthermore, the relatively low sampling frequency \( f_s = 100 \text{ kS/s} \) in contrast to the carrier signal frequency \( f = 2f_{\text{AOM}} = 160.025 \text{ MHz} \), is indicative that the proposal for a lower sampling frequency in Section 3.3 is valid.

### 4.2 Linearity of Response

The Riesz-Fischer theorem [20, 21] states that any finite time signal can be represented by a Fourier series. As such, if the phasemeter, along with the demodulation scheme, is able to accurately determine any sinusoidal relative length changes \( \Delta L \), it is reasonable to claim that the setup is able to detect all such signals.

![Figure 4.2](image)

Figure 4.2: Phasemeter was modified with a mirror on a piezoelectric stack to vary \( \Delta L \).

![Figure 4.3](image)

(a) The piezoelectric stack was set to oscillate at \( f_{\text{mirror}} = 143 \text{ Hz} \), and sampling frequency \( f_s = 1.0 \text{ MS/s} \), with carrier frequency \( f = 2f_{\text{AOM}} = 160.25 \text{ MHz} \), \( N = 4 \).

(b) Linearity of amplitude \( A \) of \( \phi_d \) with respect to increased Vpp applied to piezoelectric stack.

Figure 4.3: Figure (a) shows the optical phase difference \( \phi_d \) measured with a peak to peak voltage of 2.5 V applied to the piezoelectric stack. Figure (b) shows the linear regression of obtained amplitudes \( A \) for measured \( \phi_d \) with different peak to peak driving voltages to the piezoelectric stack.

To do so, a mirror attached to a piezoelectric stack was placed in the path of one arm of the
4.3. COMPLICATIONS OF $2\pi$ JUMPS

phasemeter, as shown in Figure 4.2. This allows us to modulate the relative length differences $\Delta L$ in the two arms of the phasemeter in a controlled manner.

A relatively low oscillation frequency was then set, and the resulting $\phi_d$ was obtained and regressed against a sinusoidal function. The obtained $\phi_d$ was then compared against the input into the piezoelectric.

In Figure 4.3, an example of obtained $\phi_d$ over 200 ms is shown in Figure 4.3a for the specified setting. A sinusoidal voltage with 2.5 peak to peak voltage at 143 Hz was applied to the piezoelectric stack. A sinusoidal regression was then performed against $\phi_d$, and the amplitude $A$ was recorded. The regression of $A$ against the various peak to peak voltages were then performed, as shown in Figure 4.3b. The error bars are too small to be seen in the figure. The linearity is therefore observed.

The same measurement was then repeated with different oscillation frequencies, and all were also found to be linear. However, the gradients $m$ in the regression were found to decrease with increased frequency of oscillation, but this was likely due to a decreasing responsiveness of the piezoelectric stack.

4.3 Complications

In the previous cases, a peculiar phenomenon was increasingly observed that could not be attributed to merely a fluke of data collection.

![Figure 4.4: A plot of (a) $\phi_d$, and (b) its corresponding radius $r$ in the I/Q-plane. This mirror oscillation was set at 1.02 V and 4.0 Hz. $f_s = 1.0$ MHz, $f_{AOM} = 80.125$ MHz.](image)

Figure 4.4a is a typically observed $\phi_d$ collected when using a short optical fibre. Looking closer, it is clear that the optical phase $\phi_d$ obtained was not natural; where the envelope of the optical phase appears to have ‘jumped’ by approximately $2\pi$ rad.
CHAPTER 4. I/Q DEMODULATION RESULTS

Figure 4.5: A closer look into the increasing $2\pi$ jump of Figure 4.4. Figure (a) and (b) provides the recorded voltage and unwrapped $\phi_d$ around the $2\pi$ jump. Figure (c) provides a plot of the individual ($I, Q$) coordinates around the occurrence of the $2\pi$ jump. Looking at points 5 and 6, it is clear that the expected phase advance of $\pi/2$ did not occur, therefore contributing to a jump.

Error Correction  Whilst investigating the reason for these jumps, a fast algorithm was created to correct them.

Figure 4.6: The error correction algorithm as described in Appendix B as applied to Figure 4.4a. The envelope appears fairly continuous, which indicates that it is indeed a jump.

The outcome of the algorithm had definitively shown that these were indeed $2\pi$ jumps, and that the envelope was indeed continuous after utilising the algorithm. As the algorithm detracts from the discussion here, it has been left for in the Appendix B. Figure 4.6 shows that the envelope is indeed continuous, and that it is therefore accurate to call these a jump.

However, we take special note that identifying the hyperparameters necessary to perform

1The algorithm runs in $O(n)$ time, where $n$ is the number of $\phi_d$ points sampled
the algorithm ideally is not a trivial matter, and that further investigation into the physical source of these jumps were necessary.

**Physical Sources of \(2\pi\) jumps** Diving deeper into the data that was collected that had produced \(\phi_d\) in Figure 4.4, was the start of investigating the source of these jumps. Initially, I had made a plot of the \((I, Q)\) coordinates around the jump as given in Figure 4.5c. Figure 4.5a, and b, are the plots of the sampled voltage \(V\) and optical phase differences \(\phi_d\) around the jump. With the \((I, Q)\) coordinates plotted, it immediately became apparent that studying just the phase, and not the radius does not paint a complete picture. This is further exemplified in Figure 4.4b, which was made with hindsight. Here, the collected data shows that the assumptions in the scheme that

1. Radius of \((I, Q)\) coordinates is fairly constant, and
2. Expected phase ‘advancement’ between consecutive points being \(\pi/2\)

were not necessarily true. As such, Figure 4.4b was made in hindsight, and it can be seen that the radius component of \((I, Q)\), i.e.: \(r = \sqrt{I^2 + Q^2}\) oscillates regularly, with values almost approaching zero.

Furthermore, using a spectrum analyzer at the photodiode signal did not reveal much that was unusual either.

The most probable explanation is that the radius approaching 0 is the source for jumps, but it still remains unclear why this would be the case.

**Increasing sampling frequency** One might consider that increasing the sampling frequency means that the difference between two consecutive optical phase difference \(\phi_i\) and \(\phi_{i+1}\) might be smaller, and therefore, the chance of a \(2\pi\) jump occurring is less.

<table>
<thead>
<tr>
<th>(f_s) / MHz</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.283</td>
</tr>
<tr>
<td>0.25</td>
<td>0.234</td>
</tr>
<tr>
<td>0.50</td>
<td>0.235</td>
</tr>
<tr>
<td>1.00</td>
<td>0.216</td>
</tr>
<tr>
<td>2.50</td>
<td>0.202</td>
</tr>
<tr>
<td>10.0</td>
<td>0.191</td>
</tr>
</tbody>
</table>

Table 4.1: Standard deviation of the distributions of \(\Delta \phi_d\) obtained for each sampled frequency \(f_s\).

Using a 130 m optical fibre, and several sampling frequencies, the histogram of difference in optical phase difference \(\Delta \phi_i = \phi_{i+1} - \phi_i\) was identified. An example for the case of 1 MHz is as shown in Figure 4.7. The results of the standard deviation for the various sampling frequencies is as shown in the corresponding table. The optical phase difference \(\phi_d\) to Figure 4.4b is shown in Figure 4.9.
While the standard deviation does decrease with increased sampling frequency, the number of samples per unit time does increase with sampling frequency. Another important point to note is that there is a sizeable number of $\Delta \phi_i$ that is almost at $\pm \pi$, which indicates numerous $2\pi$ jumps. This persists even at 10 MHz sampling frequency.

**Uncertainty evaluation**  One may consider thinking of the $(I, Q)$ coordinates as having some associated uncertainty, thereby instead allowing us to think of the coordinate as a distribution centered at some point with the average radius. The measured coordinates $(I, Q)$ is then some randomly chosen point in the distribution.

However, the radius oscillating with such regularity as exemplified in Figure 4.4b, is not a behaviour of noise, and such a consideration is to be ruled out.

Figure 4.8: Using a 130 m optical fibre instead, the demodulated optical phase difference $\phi_d$ starts to exhibit a random walk instead. $f_s = 1.0 \text{ MHz}, f_{\text{AOM}} = 70.125 \text{ MHz}$.

**Necessity for better methods of extracting phase**  Repeating the same process to obtain $\phi_d$, but now for a much longer optical fibre of 130 m in the absence of the oscillating mirror, the result is as shown in Figure 4.8 and 4.9. Here, we can see the extent of $2\pi$ jumps resemble a form of random walk, which one might have wrongly assumed as a walk in the optical fibre length, if not for the backdrop of information established up to this point.

Tuning the algorithm as described in Appendix B to correct such a random walk is unfeasible, and that a better physical sampling scheme is preferential. A better method is instead proposed in the next chapter, Chapter 5.

The next section is an attempt to try further explain the source of these jumps through the use of a mathematical model.
4.4 Mathematical Modeling

Consider the voltage that is emitted from the photodiode (as in Figure 4.2) to be of the form

\[ V(t) = V_0 \sin(2\Omega t + \phi_1) + A \sin(2\pi f_{\text{Mirror}} t + \phi_2) + \mathcal{L} + \mathcal{D}. \]  \hspace{1cm} (4.1)

Here, \( \mathcal{L}, \mathcal{D} \) denote noise terms in the length of the optical fibre, and ‘detector’ respectively.

As a toy model, consider that the fibre length noise and detector noise are that of a Gaussian noise at every time step sampled, i.e.: \( \mathcal{L} = \mathcal{N}(0, \sigma_1^2) \), and \( \mathcal{D} = \mathcal{N}(0, \sigma_2^2) \); where \( L, D \) are coupling strengths, and \( \mathcal{N}(0, \sigma_i^2) \) is the normal (Gaussian) random variable with some variance \( \sigma_i^2 \) centered at 0.

In an attempt to replicate Figure 4.4 and the signal \( V(t) \) that was sampled to create it, \( V_0 \) was set to 0.092, and \( L \) was arbitrarily set to a relatively small value of 0.020, with \( \sigma_2 = 0.3 \). Then, the value of \( L, \sigma_1 \) was slowly increased from 0, and the corresponding \( \phi_d(t) \) was then observed as shown in Figure 4.10.

If we treat the oscillating mirror \( A \sin(2\pi f_{\text{Mirror}} t) \) as the deterministic component of changes in the fibre length, and \( \mathcal{L} \) as the noise component of changes in the fibre length; it was found that for fixed small \( L, \sigma_1 \), a small value in \( A \) (i.e.: some deterministic change in the fibre length) was also necessary for \( 2\pi \) jumps to occur.

This toy model has therefore shown that deterministic changes in fibre length over time \( \Delta L(t) \), together with sufficiently noisy changes in the fibre length \( \mathcal{L} \), along with a small amount of detector noise would have created \( 2\pi \) jumps.

**Possible Improvements** We first introduce the concept of power spectral density. The spectral density function \[ S \] of frequency \( f \) (in Hertz) for a stationary signal \( x(t) \) measured
CHAPTER 4. I/Q DEMODULATION RESULTS

Figure 4.10: $2\pi$ jumps observed using the toy model. $V_0 = 0.092$, $A = 0.8$, $f_{\text{Mirror}} = 0.5$ Hz, $L = 1$, $\sigma_1 = 0.48$, $D = 0.020$, $\sigma = 0.3$, $f_s = 1.0$ MHz, $f_{\text{AOM}} = 70.125$ MHz over a finite time $T$, is given by

$$S(f) = \frac{2}{T} \left| \int_{-T/2}^{T/2} x(t) e^{-i2\pi ft} \, dt \right|^2 \quad (4.2)$$

Figure 4.11: Figure (a) is the power spectrum density (PSD) of the signal $V(t)$ as provided in the toy model to Figure 4.10. Figure (b) is the power spectrum density of the signal $V(t)$ that was sampled to have produced Figure 4.4.

Experimentally, the power spectral density of the signal that produced Figure 4.4 is shown in Figure 4.11b with a distinct peak at 250 kHz. This peak of 250 kHz is expected, as it is identically $2f_{\text{AOM}} \mod f_s$, which is the frequency at which the phase advance $\pi/4$ is expected to evolve. Note that the toy model demonstrates a distinctly white noise spectrum, whilst
the real signal deviates from a white noise spectrum, and therefore, likely to have different properties from which the toy model does not exhibit.

We however note that, the goal of this mathematical toy model is not to completely replicate the experimental data, but to give insight as to its reason for occurrence.

For the situation of the random walk as presented in Figure 4.9 one might instead consider a different noise model, such as the Weiner process \cite{22} to model the noise terms $\mathcal{L}$. The corresponding power spectrum density for the modeled $V(t)$ will be different, but does not come anywhere close to the power spectrum density of the real data.
Chapter 5

Non-I/Q Demodulation

Having no good physical reason as to the source of $2\pi$ jumps as discussed in Section 4.3, a different approach is required.

In this chapter, we will rework some of the machinery that has been discussed in Section 3.3 but using points along the sinusoidal that are not $\pi/2$ rad apart, but with any arbitrary value. This still allows for the identification of phase information of the carrier signal. This is termed as “Non-I/Q demodulation”. Thereafter, this chapter will discussed the optical phase difference $\phi_d$ obtained by non-I/Q demodulation, in contrast to the results of Chapter 4.

5.1 Digital Non-I/Q Demodulation

In Section 3.3 we stated that

$$y = A \cos \varphi_0 \cos \omega t + A \sin \varphi_0 \sin \omega t.$$  \hspace{1cm} (3.5)

Let us instead now consider measuring at times $t$ such that $\omega t$ is multiples of some $\theta$ that is a rational fraction of $2\pi$. This gives successive samples $y_j$ measured to be

$$j \in \{0, 1, \cdots, N - 1\}, \quad y_j = I \cdot \sin(j\theta) + Q \cdot \cos(j\theta).$$  \hspace{1cm} (5.1)

The value of $N$ will be further elaborated later. In the case of ideal sampling without noise,

Figure 5.1: Phasor in I/Q plane advancing by arbitary angle $\theta$. 

25
any 2 equations from Equation (5.1) would give a solution to \( I, Q \); but a different method is required in the presence of real conditions.

**Regression** To obtain the appropriate values of \( I, Q \), one can apply an unbiased estimator for the least mean squares algorithm.\(^{[23]}\) We aim to minimize the function\(^{[19]}\)

\[
f(I, Q) = \sum_{j=0}^{N-1} [I \cdot \sin(j\theta) + Q \cdot \cos(j\theta) - y_j],
\]

where \( y_j \) is the \( j \)th sampled value along the sinusoidal \( y(t_j) \). The minimization is given by\(^{[19]}\)

\[
\frac{\partial f}{\partial I} = 0, \quad \frac{\partial f}{\partial Q} = 0.
\]

Solving these would then give\(^{[19]}\)

\[
I = \frac{2}{N} \sum_{j=0}^{N-1} y_j \cdot \sin(j\theta), \quad \text{(5.4a)}
\]
\[
Q = \frac{2}{N} \sum_{j=0}^{N-1} y_j \cdot \cos(j\theta), \quad \text{(5.4b)}
\]

which are the \((I, Q)\) coordinates of the \(0\)th sampled point \( y_0 \). We can repeat the same for the next data point \( y_1 \), getting a different \((I, Q)\) coordinate, with an expected phase advance of \( \theta \).\(^{[19]}\)

### 5.2 Sampling and Carrier Frequency

This section clarifies on the values \( N \) and \( \theta \) from the previous section.

Similar to Section 3.3 consider any sampling frequency \( f_s \) and carrier frequency \( f = \omega/2\pi \),

\[
f_s = \frac{N}{M} f, \quad \text{(5.5)}
\]

but now only with requirement that \( N/M \) is positive rational \( Q > 0 \).\(^{[19]}\) Interpreting in the time domain, this means that \( N \) samples are taken in \( M \) consecutive carrier periods, such that the phasor returns to its initial position, modulo \( 2\pi \). This gives the phase advance to be

\[
\theta = \frac{M}{N} \cdot 2\pi \mod 2\pi. \quad \text{(5.6)}
\]

We take care to note that while we might not care too much for the value of \( M \), as discussed in Section 3.3, it is a logical fallacy to deduce the value of \( \theta \) directly from the value \( N \). For example, \( N = 8 \) does not imply that \( \theta = \pi/4 \), as \( \theta = 3\pi/4, 5\pi/4, 7\pi/4 \) are equally valid as a solution.
For integer $f_s$ and $f$, the values of $N, M$ are given very simply by
\begin{align}
N &= \frac{1}{\gcd(f_s, f)} f_s, \\
M &= \frac{1}{\gcd(f_s, f)} f,
\end{align}
where $\gcd(\cdot, \cdot)$ is the greatest common divisor operator.

**Phase Unwrapping**  Generalising from Section 3.4 with more detail in Appendix A.1, the relative length change (as measured through $\Delta \varphi_i$) between time $t_i$ and $t_{i+1}$ is then
\begin{align}
\Delta \varphi_i &= \begin{cases} 
(\phi_{i+1} - \phi_i - \theta) + 2\pi H (\pi + \phi_{i+1} - \phi_i - \theta), & \text{for } \theta \in (0, \pi), \\
(\phi_{i+1} - \phi_i - \theta) - 2\pi H (-\pi + \phi_{i+1} - \phi_i - \theta), & \text{for } \theta \in (-\pi, 0).
\end{cases}
\end{align}

**Sampling Scheme**  For this chapter, we chose to work with $N = 8$, and $\theta = \pi/4$. This is done by setting the sampling frequency to 1.0 MHz and $\Omega$ to 70.0625 MHz, unless otherwise specified. This is therefore, an advantage of this experimental setup, as one can very easily utilise such a different sampling scheme.

In principle, one could choose any integer for $N$, such as $N = 5$. However, the produced the optical phase difference $\phi_d$ was not sensible, and it was suspected that powers of 2 would only work. This has however, not been verified for $N = 16, 32, \cdots$.

### 5.3 Test of Concept and Resiliency Comparison

In the many repeated tests with the new scheme, no $2\pi$ jumps were initially noticed over the case of sampling in 1 s intervals. This was a significant improvement over the $N = 4$ sampling scheme. Figure 5.2 shows $\phi_d$ without deliberately generated noise, in direct comparison against Figure 4.8.

Looking closer at Figure 5.2, the envelope to the optical phase difference $\phi_d$ appears to vary much less than Figure 4.8. To further substantiate that the $2\pi$ jumps are correlated with the radius approaching 0, we note that in Figure 5.2, the radius over the entire 20 s was measured to be 0.114(14); whereas the radius over the 20 s in Figure 4.4 was measured to be 0.122(19). The decrease in standard deviation from 0.019 to 0.014 under identical physical setup, in similar noise conditions when taking $N = 4 \rightarrow 8$ means that the likelihood that it is very near 0 is much less. Both data sets were taken back to back. Therefore, jumps being related with low $r$ values is further substantiated. Furthermore, we show that increasing the value of $N$ from 4 to 8 is better at providing physically sensible optical phase difference $\phi_d$.

Taking a smaller time segment, the radius of Figure 4.8 and 5.2 is shown in Figure 5.3. One can observe the extent of this difference in standard deviation, even if the numerical difference is approximately 30%.
Figure 5.2: A plot of (a) $\phi_d$, and (b) its corresponding radius $r$ in the I/Q-plane. This was sampled without deliberate perturbation. $f_s = 1.0\text{ MHz}$, $f_{\text{AOM}} = 70.0625\text{ MHz}$.

Figure 5.3: A smaller time slice of the radius $r$ in the I/Q-plane from Figure 4.8 and Figure 5.2 respectively.
5.4 Perturbation Sensing in the Lab

One would expect that the extent of human activity to be significantly less at night than in the day. Due to the limitation of the existing setup, continuous recording of data is at present still not feasible. As such, we opted for the next best option, to record the voltage $V(t)$ as frequently as possible. As an initial run, we settled for the sampling $1 \, \text{s}$ of data every $61 \, \text{s}$.

![Figure 5.4: Standard deviation $\sigma$ of $\phi_d(t)$ over the course of one second, per 61 second interval. This was started on 24 February 2022, and was left to run for about a day. This is a proxy for the measure of human activity within the lab.](image)

After collecting the data for approximately a day, starting from 24th February 2022 15:57 pm, the optical phase difference $\phi_d$ for each time segment was then obtained, and the standard deviation $\sigma$ over the entire of $\phi_d(t)$ for the one-second interval, was obtained. The result is shown in Figure 5.4.

Comparing day against night, it is clear that the noise observed during lab hours is larger than that of at night, where people are not around in the lab, and the data conforms with our expectations.

![Figure 5.5: $\phi_d$ obtained at 5pm. This is one of the many samples taken over the course of the day.](image)
5.5 Toy Model

With reference to Section 4.4, Figure 4.10, we can replicate the mathematical modelling under identical noise conditions, but under the scheme of non-I/Q demodulation. With all values kept constant except for $f_{\text{AOM}} = 70.0625 \text{ MHz}$, the obtained $\phi_d$ is as shown in Figure 5.6.

![Figure 5.6: $2\pi$ jumps observed using the toy model. $V_0 = 0.092$, $A = 0.8$, $f_{\text{Mirror}} = 0.5 \text{ Hz}$, $L = 1$, $\sigma_1 = 0.48$, $D = 0.020$, $\sigma = 0.3$, $f_s = 1.0 \text{ MHz}$, $f_{\text{AOM}} = 70.0625 \text{ MHz}$](image)

This further supports the robustness of $N = 8$ non-I/Q demodulation, in contrast to the case of $N = 4$ for I/Q demodulation.
Chapter 6

Summary

This thesis, first demonstrates the modification of a typical fibre-based Michelson interferometer into a heterodyned phasemeter. Several properties of our experimental setup was then characterised, before the I/Q demodulation scheme was described. This I/Q demodulation scheme allows for the digital sampling of data from the phasemeter to determine the directional displacement in the optical path difference between the two arms of the interferometer.

With the I/Q demodulation scheme, experimental results were collected and analysed, but an issue with jumps was soon found. Work was then placed into the correction, and identification of the source of this jumps, but no good physical reason could be found.

As an advantage of the experimental setup, non-I/Q demodulation was explored, and the results proved successful in contrast to I/Q demodulation.
Appendix A

Optimisations

A.1 Phase “unfolding”

In Chapter 3.4, the “unfolding” of measured absolute phase $\phi_i \in (-\pi, \pi]$ to optical phase difference $\phi_d \in \mathbb{R}$ was shown. For ease of reference, the key equations are repeated here:

$$\phi_{i+1} - \phi_i \approx \frac{\pi}{2}$$  \hspace{1cm} \text{(3.7)}

$$\Delta \phi_i = \begin{cases} \phi_{i+1} - \phi_i - \frac{\pi}{2} & \text{if } |\phi_{i+1} - \phi_i - \frac{\pi}{2}| < |\phi_{i+1} - \phi_i - \frac{\pi}{2} + 2\pi|, \\ \phi_{i+1} - \phi_i - \frac{\pi}{2} + 2\pi & \text{else}. \end{cases}$$  \hspace{1cm} \text{(3.9)}

which was said to required optimisation. As further shown in Chapter 5, Equation (3.7) could have instead been any sensible value, such as between 0 and $\pi$.

Since $\phi_i \in (-\pi, \pi]$, consider a variable $u$ defined by $\phi_{i+1} - \phi_i \in (-2\pi, 2\pi]$, and variable $\theta \in (0, \pi]$, which plays the role of generalising the $\pi/2$ in Equation (3.7). Studying Equation (3.7) closely, we notice that $u$ and $\theta$ always come together, and therefore we define $x = u - \theta \in (-2\pi - \theta, 2\pi - \theta]$.

We utilise a concept in computer science known as branchless programming, where conditional statements such as if, switch and other conditional statements translates to a mov command in Assembly, among other inefficiencies. Removing avoidable branches is therefore a good practice.

This encourages us to define a mathematical function that does not require the use of a ‘branch’, and with some experimentation, we get the function $f$ defined by

$$\Delta \phi = f(x) = f(u - \theta) = x + 2\pi H(\pi + x),$$  \hspace{1cm} \text{(A.1)}

where $H(x)$ is the Heaviside function. Plotting for fixed values of $\theta$, we get Figure A.1, which behaves as we expect.

That said, while it has not been rigorously tested with the experimental setup, it is not adviseable to choose values of $\theta$ too close to $\pi$.

The next part is written in context of Section 5.1, where the possible values of $\theta$ could be much different. For choices of $\theta \in [\pi, 2\pi]$ (which is equivalent to $\theta \in [-\pi, 0)$ by the $2\pi$ periodicity), note that the intuition that led to Equation 3.9 would differ slightly. The phase
Figure A.1: A plot of expected $\Delta \phi_i$ for various $x$, with $\theta$ fixed to 4 values.

Advance would then have one expect that $\Delta \phi < 0$, and the jump as shown in Figure 3.7 would instead be

$$
(\phi_{i+1} - \Delta \phi_i) - (\phi_i + \theta) = \begin{cases} 
2\pi & \text{if } \phi_{i+1} \text{ “crosses below” } -\pi, \\
0 & \text{if } \phi_{i+1} \text{ does not “cross below” } -\pi.
\end{cases} \tag{A.2}
$$

Rewriting $\Delta \phi_i$ as the subject, we can say that a relative length change (as measured through $\Delta \phi_i$) between time $t_i$ and $t_{i+1}$ to be

$$
\Delta \phi_i = \begin{cases} 
\phi_{i+1} - \phi_i - \theta & \text{if } |\phi_{i+1} - \phi_i - \theta| < |\phi_{i+1} - \phi_i - \theta - 2\pi|, \\
\phi_{i+1} - \phi_i - \theta - 2\pi & \text{else.}
\end{cases} \tag{A.3}
$$

The branchless function $f$ is then written as

$$
\Delta \phi = f(x) = f(u - \theta) = x - 2\pi H(-\pi + x) \tag{A.4}
$$

for $\theta \in [-\pi, 0)$. 
A.2 Discrete Convolution

Mathematical Definition  For complex-valued functions $f, g$ defined on the set $\mathbb{Z}$ of integers, the discrete convolution of $f$ and $g$ is given by

\[
\{f * g\}(n) = \sum_{m \in \mathbb{Z}} f(m)g(n-m) \tag{A.5a}
\]

or equivalently

\[
\{f * g\}(n) = \sum_{m \in \mathbb{Z}} f(n-m)g(m). \tag{A.5b}
\]

Example of Convolution Used in the Project  With reference to Section 5.1, the following equations are reproduced here.

\[
I = \frac{2}{N} \sum_{j=0}^{N-1} y_j \cdot \sin(j\theta), \tag{5.4a}
\]

\[
Q = \frac{2}{N} \sum_{j=0}^{N-1} y_j \cdot \cos(j\theta). \tag{5.4b}
\]

This is a strong example where convolution is used a lot in the project. Utilising a for loop in any high level, interpreted language such as Python would be very slow. Optimising this step to utilise convolutions, which often has native or optimised open-source implementations, would be ideal.

We recall that we wish to determine $(I_i, Q_i)$ for the sampled points $y_i$, for which there are a lot of. Looking closely at the two equations, it looks similar to Equation A.5a with a scale factor, and the domain $\mathbb{Z}$ has been constrained. Properly rewriting for all sampled points $y_i$ and their corresponding $(I_i, Q_i)$ coordinate, we have

\[
I_i = \frac{2}{N} \sum_{m=i}^{N-1+i} y_m \cdot \sin((m-i)\theta), \tag{A.7}
\]

\[
Q_i = \frac{2}{N} \sum_{m=i}^{N-1+i} y_m \cdot \cos((m-i)\theta). \tag{A.8}
\]

This can be easily be written as a convolution, by identifying $f$ with $y$, and $g$ by

\[
i \in \{0, 1, \ldots, N-1\}, \quad g(i) = \sin((N - 1 - i)\theta),
\]

or the cos equivalent to obtain $Q$. Note that if there are $l$ sampled values of $y_i$, then the last $N-1$ points do not have their coordinates known. In other words, only $l + 1 - N$ points will have $I, Q$ coordinates known.
Example in Python  Using numpy np, we first define a sine and cosine “lookup table” by

```python
from numpy import pi
N = 8  # Example
theta = 2*pi/N  # Example
sines = np.sin([i * theta for i in range(N)])
cosines = np.cos([i * theta for i in range(N)])
```

which can then use for convolution. Defining the values of y by `signal`, we can get the $I, Q$ coordinates by

```python
Is = (2/N) * np.convolve(signal, sines[::-1], mode = 'valid')
Qs = (2/N) * np.convolve(signal, cosines[::-1], mode = 'valid')
```

The use of `[::-1]` is by definition of the convolution operator (and likewise, the slightly ugly definition of $g$).

This is many orders of magnitude faster than using the `for` loop within Python.

Example in GNU Octave  We first define a sine and cosine “lookup table” by

```octave
sines = sin([0:N-1] * theta)
cosines = cos([0:N-1] * theta)
```

which can then be used for convolution. Defining the values of y by `signal`, we can get the $I, Q$ coordinates by

```octave
Is = (2/N)*conv(signal, fliplr(sines), "same")
Qs = (2/N)*conv(signal, fliplr(cosines), "same")
```

Example in Mathematica  We first define a sine “lookup table” by

```mathematica
Nv = 8 (*Example*)
theta = 2*Pi/Nv (*Example*)
sines = Sin[(Range[Nv]-1)*theta]
cosines = Cos[(Range[Nv]-1)*theta]
```

which can then be used for convolution. Defining the values of y by `signal`, we can get the $I, Q$ coordinates by

```mathematica
Is = ListConvolve[Reverse[sines], signal]
Qs = ListConvolve[Reverse[cosines], signal]
```

For reference, I also note that another strong example of convolution used is in obtaining $\Delta \phi_i$ in Equation (A.3).
Appendix B

2π Jumps Error Correction Algorithm

This appendix is written to supplement a tangential discussion on work that was done in the process of the project, as aforementioned in Section 4.3. In this section, we let \( \phi_i \) be the individual timesteps \( t_i \) of the demodulated optical phase difference \( \phi_d(t_i) \), rather than the argument of the I, Q coordinate, as given by \( \tan^{-1}(Q/I) \)—the measured phase.

Initial Attempt to Error Correction Algorithm When taking a closer look at the \( \phi_d \) data generated (such as in Figure 4.4a), a recurring pattern one quickly observes is that the phase is likely to have a \( 2\pi \) jump when

\[
|\phi_{i+1} - \phi_i| \gg 0,
\]

where \( \pi/2 \) plays the role of expected phase change here, such in the case of \( N = 4 \), as presented in the context of Section 4.3. In principle, this algorithm can also be extended to the case of Chapter 5 if necessary, but will not be discussed here. To formalize the notion of \( \gg 0 \) as presented in the above equation, one can take a look at the distribution of \( \phi_{i+1} \) with the phase advancements of \( \phi_i \) to \( \phi_{i+1} \) with

\[
|\phi_{i+1} - \phi_i| > \delta
\]

is considered anomalous/‘large’. Successive anomalous phase advancement of the same sign are then added up until the phase advancements returns to being non-anomalous. If the total sum is approximately \( 2\pi \), the algorithm can then shift the all subsequent \( \phi_d \) by appropriate multiples of \( 2\pi \). We denote the number of consecutively anomalous phase increments (or decrements) by \( n \).

Issues with initial attempt This algorithm has several hyperparameters of arbitrary choice to nail down.

First, there are two parameters \( \epsilon_{\pm} \) associated to the phrase “approximately \( 2\pi \)”. Next there are 2 parameters to bound the expected values of \( n \) from below and above, which are also
implicitly variable with $\delta$. Very often, a strict choice (i.e. large) of $\delta$ would have meant that a $2\pi$ jump might have occurred with just $n = 2$, which is a very low threshold for $n$, and also required the parameters $\epsilon_\pm$ to have been much larger.

Alternatively, one could relax the tolerance $\delta$, but the value of $n$ could have then even raised to 5, and a stricter to choice of $\epsilon_\pm$ would have been harder to have narrowed down on.

While it is entirely possible to perform some kind of machine learning (probably, a grid search from gut instinct) to narrow down on the appropriate choice of parameters, such effort would prove to be wasteful with just some thought, as the noise exhibited in $\phi_d$ has been shown to vary with the environment, and even the length of fibre, where the choice of parameters would likely need to change.

As such, this algorithm shall be left in this appendix in words, and no code is replicated here. We now present a slightly wiser algorithm.

**Second attempt to the Error Correction Algorithm** In the case of $N = 4$, as prior mentioned, one does not expect $n \geq 8$ for reasonable choices of $\delta$, even if small. As such, we define a ‘difference window’ $d = 8$, and a cache of length $8d$. The difference window is done across the cache. In the difference window, a $2\pi$ jump has occurred if

1. Within any given window, there is indeed a jump of approximately $2\pi$, i.e.:
   
   $$|\phi_{i+d} - \phi_i| \approx 2\pi,$$

2. Within the cache but outside the found window, the phase is relatively constant.

The second condition is necessary, as it is physically possible for consecutive $2\pi$ shift over all windows within the cache. This removes away the arbitrary choice of $\delta$, but the idea of when a change is large enough to start accounting for consecutive sign difference is still useful.

The figure in the following flowchart gives a pictorial representation of what things look like during a jump.

![Flowchart of idea to tackling $2\pi$ jumps.](image)

Due to the constraint to keeping as minimal computations as possible, alot of the code is done in a convoluted manner. The code as in python, is as follows.
```python
import numpy as np

from math import copysign

sign = lambda x: 0 if x == 0 else int(copysign(1,x))

class fix2pi:
    def __init__(self):
        self.stream = []
        self.shift = 0
        self.shift_counter = 0
        self.prevjump = 0
        self buffering = 0
        self.cutoff = 64
        self.rolen = 8
        self.rolavg = []
        self.counter = 0
        self.tol = 1 # 2 * rms(noise) + a bit more

    def process(self, data):
        # main loop that takes in
        data += self.shift*2*pi
        self.stream.append(data)
        self.counter += 1

        # when buffer is not filled, populate and terminate fn call
        # move data from stream into rolav
        if self.buflen != self.cutoff:
            self.buflen += 1
            if self.buflen == self.rolen:
                self.rolavg.append(sum(self.stream)/self.rolen)
            elif self.buflen >= self.rolen:
                self.rolavg.append( \n                    (self.stream[-1]-self.stream[-1-self.rolen]) / self.rolen+ self.rolavg[-1] )

        return None

        # addition of 2 numbers to get rolling average
        self.rolavg.append( \n            (self.stream[-1]-self.stream[-1-self.rolen]) / self.rolen+ self.rolavg[-1] )

        # check if jump is centered around 2 pi
        jump = self.rolavg[-1] - self.rolavg.pop(0)
        jump_sub = abs(jump) - 2*np.pi
        if abs(jump_sub) < self.tol:
            # check the signs are the same
            if sign(jump) == self.prevjump:
                self.shift_counter += 1
            else:
                self.shift_counter = max(0, self.shift_counter-1)
        else:
            self.shift_counter = max(0, self.shift_counter-1)

        self.prevjump = sign(jump)
```

The code is mostly self-explanatory.

The optimisations done are to utilise a rolling average to reduce the extent of noise in the phases (denoted as `stream`). The rolling average is done this way to avoid the need to recomputed over all of `rollen` (“roll length”) each and every time a new phase is piped in. After performing `jump = self.rolavg[-1] - self.rolavg.pop(0)`, there exists a small spike into $2\pi$ with some width if there has been a $2\pi$ jump. If the correct number of consecutive sign changes have occurred concurrently (i.e. the width is appropriate), as given by `shift_counter`, the $2\pi$ fix is invoked. The point within the array is manually pointed out for the rising and falling edges.

However, there are still some parameters that are fairly arbitrary, such as the 23, and the noise tolerance $\epsilon$. As reiterated from the initial attempt, whilst it may be possible to converge on the appropriate hyperparameters, it is likely to vary between various noise situations, and therefore unlikely to be practical.


22Kurt Jacobs, Stochastic processes for physicists: understanding noisy systems (Cambridge University Press, 2010).