(Incomplete) Lecture notes for PC5201S

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Chapter 1

Time-discrete signals

Time-discrete signals are typically used to approximate time-continuous signals, and are commonly found in modern digital signal handling.

Typically, a time-continuous signal $f(t)$ is sampled at discrete, equally spaced times t that are separated by a fixed sampling interval Δt:

$$f(t) \rightarrow f_k = f(t = k\Delta t)\,, \quad k \in \mathbb{Z}\,. \tag{1.1}$$

The signal is now represented by a discrete sequence of numbers f_k, that map the original function in are "reasonable" manner. Intuitively, if the interval Δt is chosen small enough, the features in the original signal $f(t)$ is captured sufficiently well. The sequence of numbers f_k is an infinite set of numbers, in the same way as the time argument t in the original function is not limited.

1.1 Sampling and the Dirac comb function

A useful concept to formalize this transition from a continuous to a sampled signal makes use of an array of delta functions spaced in equal time intervals Δt. We refer to this as

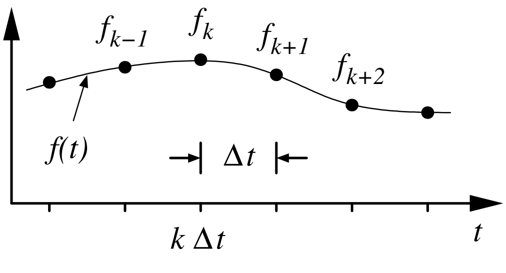

Figure 1.1: A time-continuous function $f(t)$ gets sampled at discrete times $t = k\Delta t$, leading to samples f_k.

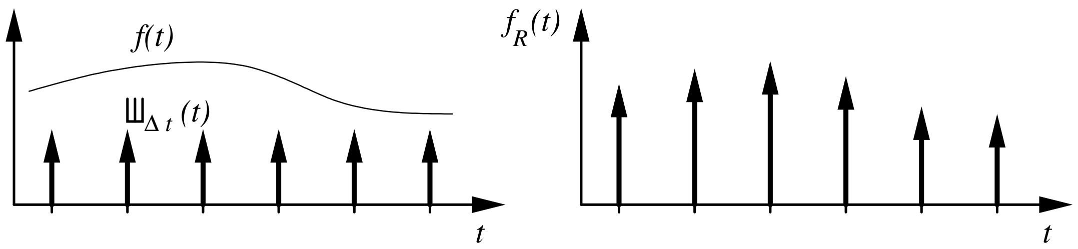

Figure 1.2: Multiplication of the original function $f(t)$ with a Dirac comb $Ш\Delta t(t)$ leads to a representation $f_R(t)$ of the sampled function that can be used to understand its spectral properties.

the Dirac comb function

$$Ш_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t)\,. \tag{1.2}$$

To represent the time-discrete sampled signal in a time-continuous way, we can use a similar concept to the comb function above, but modify the weight of each δ peak. This can be done by simply multiplying $f(t)$ by the comb function $Ш_{\Delta t}(t)$,

$$f_R(t) = f(t) \cdot Ш_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - k\Delta t)\,, \tag{1.3}$$

as the Kronecker-δs in $Ш_{\Delta t}(t)$ sample the original function at the correct positions (see Fig. 1.2).

This new function $f_R(t)$ has, for $T \gg \Delta t$ and for a sufficiently smooth original function $f(t)$, approximately the same integral over T as the original function:

$$\int_0^T f(t)dt \approx \int_0^T f_R(t)dt = \int_0^T \sum_{k=-\infty}^{\infty} f_k \delta(t - k\Delta t)dt \approx \Delta t \sum_{k=0}^{T/\Delta t} f_k\,. \tag{1.4}$$

1.1.1 Spectral properties of sampled functions

To understand quantitatively the effect of sampling, it is helpful to compare the spectral content of $f_R(t)$ to the one of the continuous signal $f(t)$, given by the Fourier-transformed signal:

$$\tilde{f}(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt\,. \tag{1.5}$$

The Fourier transformation of $f_R(t)$ can then be calculated (see details in appendix A.2):

$$\tilde{f}_R(\omega) = \sum_{k=-\infty}^{\infty} f_k\, e^{-i\omega\Delta t \cdot k}\,, \tag{1.6}$$

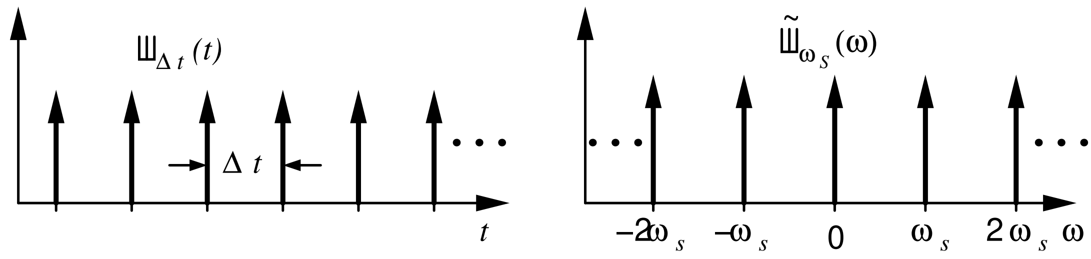

Figure 1.3: A Dirac comb $\text{Ш}_{\Delta t}(t)$ in time with a separation of Δt between the peaks has a frequency spectrum that is also a Dirac comb, with a frequency separation ω_S between the different components.

This is a *continuous* function in ω, even though the underlying time series f_k is discrete; this is referred to as a Fourier series. As such, $\tilde{f}(\omega)$ is a periodic function:

$$\tilde{f}(\omega) = \tilde{f}(\omega + k \cdot 2\pi/(\Delta t)) \quad \text{for} \quad k \in \mathbb{Z}\,. \tag{1.7}$$

A special case of such a Fourier series is one where $f_k = 1$ for all k, corresponding to $f_R(t) = \text{Ш}_{\Delta t}(t)$, the Dirac comb. In this case, the result is

$$\mathcal{F}\left[\text{Ш}_{\Delta t}(t)\right] = \sum_{k=-\infty}^{\infty} e^{-i\omega\Delta t \cdot k} \tag{1.8}$$

The Dirac comb function can be represented by a Fourier series,

$$\text{Ш}_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} e^{i\frac{2\pi}{\Delta t}nt}\,, \tag{1.9}$$

so the Fourier transform of the Dirac comb function is again a Dirac comb. Proper care of pre-factors (see appendix B) gives

$$\mathcal{F}[\text{Ш}_{\Delta t}(t)] = \omega_S \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S) \quad \text{with} \quad \omega_S = \frac{2\pi}{\Delta t}\,. \tag{1.10}$$

This allows for a useful interpretation of the spectral content of a sampled system presented in (1.6), making use of convolution theorem (C.2):

$$\begin{aligned}
\tilde{f}_R(\omega) &= \mathcal{F}[f(t) \cdot \text{Ш}_{\Delta t}(t)] = \tilde{f}(\omega) * \mathcal{F}[\text{Ш}_{\Delta t}(t)](\omega) \\
&= \int_{-\infty}^{\infty} \tilde{f}(\omega') \cdot \mathcal{F}[\text{Ш}_{\Delta t} t)](\omega - \omega') d\omega' \\
&= \int_{-\infty}^{\infty} \tilde{f}(\omega') \left[\omega_S \sum_{k=-\infty}^{\infty} \delta(\omega - \omega' - k\omega_S)\right] d\omega' \\
&= \omega_S \sum_{k=-\infty}^{\infty} \tilde{f}(\omega - k\omega_S)
\end{aligned} \tag{1.11}$$

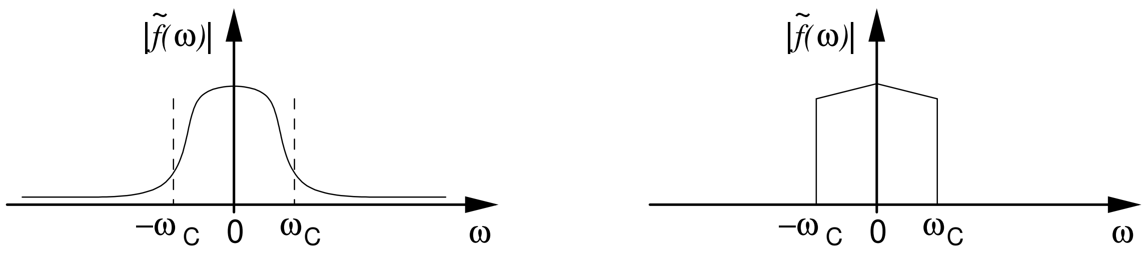

Figure 1.4: Fourier distributions $\tilde{f}(\omega)$ (or spectra) of typical signals, with a representative cutoff frequency ω_C, indicating a region $-\omega_C \ldots \omega_C$ in which most of the signal is contained. The left sample indicates a realistic spectrum, the right side a symbolic distribution as often used when representing signals. The distributions are symmetric around $\omega = 0$, a property of all real-valued signals $f(t)$. Both distributions would be a assigned a bandwidth $B = \omega_c/2\pi$ (when referenced in frequencies rather than angular frequencies).

This means that the spectrum $\tilde{f}_R(\omega)$ of a (reconstructed) sampled signal contains a superposition of copies the original signal spectrum, $\tilde{f}(\omega)$, shifted by integer multiples of the sampling (angular) frequency ω_S. This has two important consequences, namely a relationship between useful bandwidth of a signal and the sampling frequency, and a method of shifting frequency components simply by sampling.

1.1.2 Sampling theorem

We first return to the intuitive insight that, in order to capture the essential features of a continuous signal $f(t)$ in a sampling process, the sampling time interval Δt has to be significantly shorter than the time scale of changes in the original signal $f(t)$. To quantify the smoothness of the original signal, we assume that the spectral amplitudes $\tilde{f}(\omega)$ decrease significantly above a certain frequency ω_C, referred to as a cutoff frequency or the bandwidth of a signal. Such a situation is e.g. found in the range of audible signals, where the bandwidth corresponds to a few kHz, but just about any signal that is processed in some way has such an upper frequency ω_C.

We now consider a situation where a bandwidth-limited signal $f(t)$ is sampled with a time interval Δt. The condition of sampling with a sufficiently high rate compared to the time scale of changes in $f(t)$ is reflected in a condition $\omega_S \gg \omega_C$, where $\omega_S = 2\pi/\Delta t$ is the sampling (angular) frequency, and ω_C a cutoff frequency (or bandwidth) of the bandwidth-limited signal.

The resulting spectrum of the reconstructed sampled signal $f_R(t)$ through (1.11) now is a sum of copies of the original spectrum $\tilde{f}(\omega)$, separated by multiples of the sampling frequency ω_S (see Figure 1.5).

To reconstruct the original spectrum $\tilde{f}(\omega)$ from $\tilde{f}_R(\omega)$, a low pass filter can be applied to the sampled spectrum to remove the higher order copies of the baseband spectrum. This works as long as spectral components of the first order copy around ω_S does not

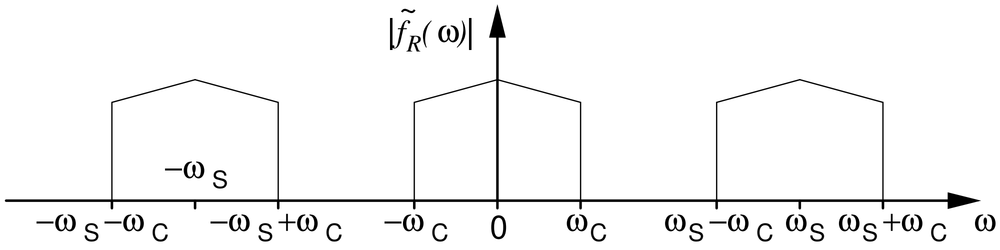

Figure 1.5: Spectrum $\tilde{f}_R(\omega)$ of the reconstructed sampled signal, sampled with ω_S. Copies of the baseband spectrum $\tilde{f}(\omega)$ appear at integer multiples of ω_S.

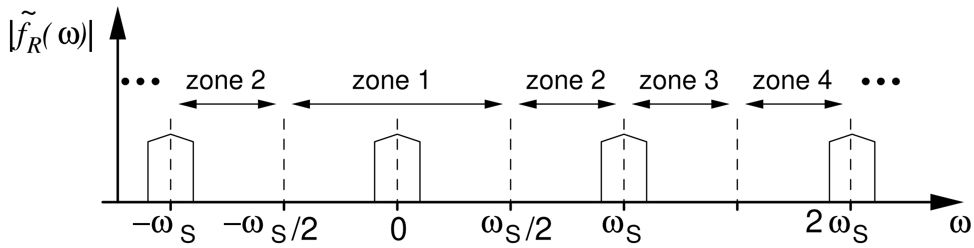

Figure 1.6: Segmentation of a spectrum in Nyquist zones spaced by half the sampling frequency ω_S. Copies of the original spectrum appear in the odd zones, whereas copies of mirrored original spectra appear in the even zones.

overlap with the baseband spectrum. Quantitatively, this means that

$$\omega_C < \omega_S/2\,, \tag{1.12}$$

or in other words, the highest frequency ω_C in the signal has to be smaller than half the sampling frequency ω_S in order to be uniquely reconstructible. This relationship is referred to as the *sampling theorem* or *Nyquist theorem*. An example is the time-discrete sampling of an audio signal in the CD standard, which uses $\omega_S = 2\pi \cdot 44.1\,\text{kHz}$, which in theory would be able to reproduce audio signals with an upper frequency around 20 kHz, the perception limit of most (young) humans.

1.1.3 Nyquist zones

For a given sampling frequency ω_S, the whole spectrum can be segmented into different zones, separated by multiples of half the sampling frequency (see Figure 1.6). Then, the sampling theorem states that in order for a sampled signal to be uniquely reconstructible, its spectrum has to reside in the first "Nyquist" zone. Higher order zones, e.g. the second zone covering the interval $[\omega_S/2, \omega_S]$ and its negative counterpart $[-\omega_S, -\omega_S/2]$ contain copies of the original signal spectrum. For even zones, they contain the mirrored original spectrum, while the odd zones contain the original spectrum.

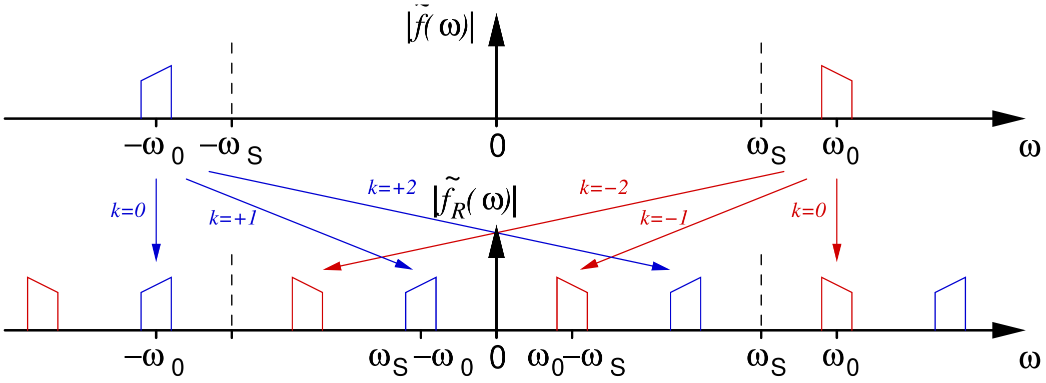

Figure 1.7: A signal with a spectrum centered around a frequency ω_0, when sampled with frequency ω_S, leads to a reconstructed spectrum $|\tilde{f}_R(\omega)|$ with a frequency spectrum shifted by multiples of ω_S. In the example shown above, the spectrum centered around ω_0 in the 3. Nyquist zone gets moved to a new center frequency $\omega_0 - \omega_S$ in the first zone.

1.1.4 Frequency shifting by sampling

The appearance of several copies of the original spectrum $\tilde{f}(\omega)$ in a sampled signal at multiples of the sampling frequency can be conveniently used to shift the signal of a spectral band. Modern software-defined radio receivers and similar signal processors make extensively use of this idea, as a narrow spectral segment at a high carrier frequency can be converted to a lower frequency band by adequate sampling.

If the spectrum of a real-valued signal $f(t)$ is contained in a small band around a carrier frequency ω_0, its Fourier transform is actually non-zero in two bands at positive and negative frequencies. The sampling process then generates multiple copies of both bands at positive and negative frequencies (see figure 1.7).

Now, the condition for unique reconstructability requires that the signal has to be contained in a single Nyquist zone to avoid overlap with other mirrored or displaced copies of the spectrum generated in the sampling process.

A restriction of the reconstructed sampled signal to the first Nyquist zone, as an appropriate filtering of a time-discrete filtering will provide, will lead to a spectrum of the reconstructed sampled signal $\tilde{f}_R(\omega)$ that is frequency-shifted from its original center frequency by multiples of the sampling frequency.

When the shifting is from an even Nyquist zone to the first one, the frequency content gets mirrored. There, higher frequency components in the sampled signal closer to a multiple of the sampling frequency will be transferred closer to frequency 0, i.e., to lower frequencies. When the shifting is from an odd Nyquist zone to the first zone, the complete spectrum is just shifted by multiples of ω_S.

To come back to the practical application of a higher-order Nyquist sampling in a software-defined radio, the sampling of the original signal should happen in a way that is well-represented by a multiplication of the original signal by a Dirac δ function according to (1.3). In practice, this means that the sampling mechanism should be significantly shorter than the sampling interval Δt.

A consequence of this process is that, if a time-continuous signal $f(t)$ has spectral components in higher Nyquist zones, their spectral content will be mapped into the first Nyquist zone, and interfere with the spectral content there. Therefore, contributions form other Nyquist zones than that of interest need to be suppressed before sampling.

1.2 Time-discrete filters

Filters are among the most important signal processing elements. To formally describe these elements for time-discrete signals, it is useful to recall how conventional (i.e., time-continuous) filters are described. Filters transform a signal $f(t)$ into another signal $g(t)$. They are are typically time-invariant in their action,

$$f(t+T) \to g(t+T) \quad \text{for all} \quad T\,, \tag{1.13}$$

linear in their response,

$$a \cdot f(t) \to a \cdot g(t)\,, \tag{1.14}$$

which implies that sums of signals are transformed into the sum of their filtered versions, and can most conveniently described as a multiplication with a frequency-dependent gain $h(\omega)$ in the Fourier domain:

$$\tilde{g}(\omega) = \tilde{h}(\omega) \cdot \tilde{f}(\omega)\,. \tag{1.15}$$

Using the convolution theorem, the filter action can also be written as

$$g(t) = h(t) * f(t) = \int\limits_{-\infty}^{\infty} h(t-t') \cdot f(t')dt'\,, \tag{1.16}$$

with $h(t) = \mathcal{F}^{-1}[\tilde{h}(\omega)]$ referred to the *impulse response* of the filter. It is also referred to as the *Green function* of a filter.

Even for real-valued signals $f(t)$ leading to real-valued results $g(t)$, the filter gain $\tilde{h}(\omega)$ can be complex-valued. However, to maintain a real-valued output $g(t)$ from a real-valued input $f(t)$ in time domain, the filter function must must obey

$$\tilde{h}^*(\omega) = \tilde{h}(-\omega)\,. \tag{1.17}$$

Another common property of filters implemented in real time is *causality*, namely, that the response of the filter at any given time can not depend on signals in the future, the impulse response can only have contributions for positive times:

$$h(t) = 0 \quad \text{for} \quad t < 0\,. \tag{1.18}$$

To describe filters for time-discrete signals, one one can simply replace the time-continuous signals $f(t)$ and $g(t)$ in the expressions above with weighted Dirac combs (1.3).

As the Fourier transforms $\tilde{f}(\omega), \tilde{g}(\omega)$ are then periodic in ω, the filter gain $\tilde{h}(\omega)$ must be as well,

$$\tilde{h}(\omega + k\omega_S) = \tilde{h}(\omega), \quad k \in \mathbb{Z}, \tag{1.19}$$

but the filter action description (1.15) remains unchanged. The periodicity of $\tilde{h}(\omega)$ implies that the impulse response $h(t)$ of the filter can also represented by a time-discrete stream h_k,

$$h(t) = \sum_{k=-\infty}^{\infty} h_k \delta(t - k\Delta t). \tag{1.20}$$

In time domain, the action of a filter on a can be described as

$$\begin{aligned}
g(t) &= h(t) * f(t) = \int_{-\infty}^{\infty} h(t - t') \cdot f(t') dt' \\
&= \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} h_k \delta(t - t' - k\Delta t) \right] \cdot \left[\sum_{l=-\infty}^{\infty} f_l \delta(t' - l\Delta t) \right] dt' \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_k f_l \delta(t - (l + k)\Delta t) \\
&= \sum_{k=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} h_k f_{u-k} \delta(t - u\Delta t) \\
&= \sum_{u=-\infty}^{\infty} \underbrace{\sum_{k=-\infty}^{\infty} h_k f_{u-k}}_{=: g_u} \delta(t - u\Delta t).
\end{aligned} \tag{1.21}$$

So the convolution for time-discrete signals takes simply the form of a sum,

$$g_k = \sum_{l=-\infty}^{\infty} h_l f_{k-l}. \tag{1.22}$$

For causal filters, $h(l < 0) = 0$, so that the summation only need to be carried out over positive l.

This form also suggests to distinguish two classes of filters. If the sum in (1.22) has to be carried over all l, the filter shows an *infinite impulse response*. If coefficients h_l vanish for some index $l > L$, the filter has a *finite impulse response*.

1.2.1 Time delay blocks

A basic element for implementing time-discrete filters is a storage element. It is typically implemented by a register, clocked by the sampling frequency. It's action on the discrete input stream f_k can be described as

$$g_k = f_{k-1}. \tag{1.23}$$

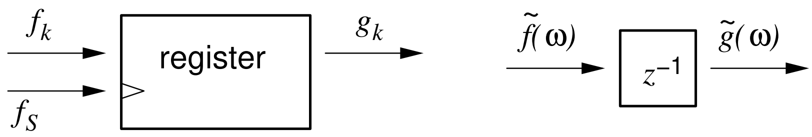

Figure 1.8: Storage element, implemented by a single register that stores a discrete signal f_k each sampling cycle, leading to a delayed copy g_k. This can be represented in Fourier domain by multiplication with a complex value z^{-1}.

Its action in Fourier domain can simply be calculated from this defining expression:

$$\begin{aligned}\tilde{g}(\omega) &= \sum_{k-=-\infty}^{\infty} g_k e^{-i\omega\Delta tk} = \sum_{k-=-\infty}^{\infty} f_{k-1} e^{-i\omega\Delta tk} \\ &= \sum_{k-=-\infty}^{\infty} f_{k-1} e^{-i\omega\Delta t(k-1)} e^{-i\omega\Delta t} \\ &= z^{-1} \cdot \tilde{f}(\omega)\,, \quad \text{with} \quad z = e^{i\omega\Delta t} \end{aligned} \tag{1.24}$$

A time delay by Δt, here one sampling cycle, can be represented by a filter function $\tilde{h}(\omega) = z^{-1}$, a pure phase factor with a phase linear in Δt. This is exactly the same filter function that represents a time delay in time-continuous signals. As the corresponding impulse response in time domain is

$$h_k = \delta_{k,1} \tag{1.25}$$

the delay element belongs to the class of finite impulse response filters.

1.2.2 Floating average filter

Another building block consists of a floating average filter. For an averaging over N samples, its formal defining action can be written as

$$g_k = \frac{1}{N} \sum_{n=0}^{N-1} f_{k-n}\,, \tag{1.26}$$

corresponding to its impulse response function

$$h_k = \begin{cases} 1/N, & 0 \le k < N\,, \\ 0 & \text{otherwise.} \end{cases} \tag{1.27}$$

The frequency-dependent filter gain $\tilde{h}(\omega)$ can be directly calculated from this by Fourier transformation,

$$\tilde{h}(\omega) = \sum_{n=0}^{N-1} \frac{1}{N} e^{i\omega\Delta tn} = \frac{1}{N} \frac{1 - e^{-i\omega\Delta t \cdot N}}{1 - e^{-i\omega\Delta t}}\,, \tag{1.28}$$

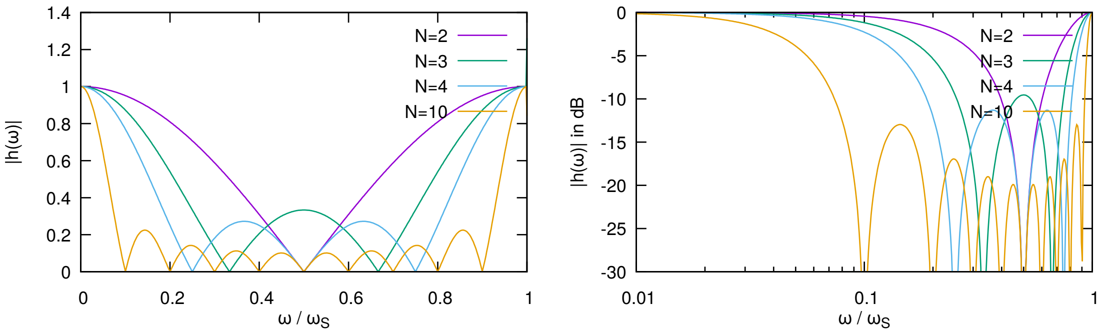

Figure 1.9: Filter transmission of a moving average filter with N averaging points for a sampled signal, in linear and logarithmic scales. The filter transmission has several zeros, and is symmetric in the first and second Nyquist zone.

were the key step is simply using an expression for a geometric sum. It is instructive to re-write this expression:

$$\tilde{h}(\omega) = \frac{1}{N} e^{-i\omega\Delta t(N-1)/2} \frac{\sin(\omega \Delta T N/2)}{\sin(\omega \Delta T/2)} \,. \tag{1.29}$$

A global phase factor, linear in frequency, corresponding to a time delay of half of the averaging time, followed by a real-valued ration of two sine functions in frequency. The latter provides strict zeros at frequencies $\omega_z = 2\pi k/(N \cdot \Delta t)$ for $k > 0$. In the limit of $\Delta t \to 0$, the latter ratio converges against a sinc function, as expected for the Fourier transformation of a square pulse. The frequency response of the modulus of the filter is shown in figure 1.9. The filter gain $\tilde{h}(\omega)$ has the characteristic of a low pass filter, reflecting the intuitive impression that averaging over several samples suppresses faster changes in the sampled signal.

Using the notation $z = e^{i\omega\Delta t}$, the transfer function can also be written in a compact form that is often found in signal processing literature:

$$\tilde{h}(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} z^{-n} = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}} \tag{1.30}$$

In principle, the discrete coefficients h_k of the impulse response of a filter with a desired frequency-dependent gain $\tilde{h}(\omega)$ could be obtained by an inverse Fourier transformation of the periodic $\tilde{h}(\omega)$:

$$h_k = \frac{1}{2\pi} \int_{\omega_S/2}^{\omega_S/2} \tilde{h}(\omega) e^{ik\omega\Delta t} \,, \quad \text{for} \quad k \in \mathbb{Z} \,. \tag{1.31}$$

However, such a filter is not automatically causal, nor has it a finite impulse response. Even the standardized filters (see ...) have an infinite response.

1.2.3 Filters with finite impulse response (FIR)

Implementing a filter numerically through direct convolution of a signal with a impulse response through (1.22) requires a finite length L,

$$g_k = \sum_{l=-\infty}^{L-1} h_l f_{k-l} \,. \tag{1.32}$$

Finding a finite set $\{h_l\}$ for a desired filter characteristic is a complex process, and can often be only solved numerically and approximately.

One consideration is that coefficients h_l obtained by (1.31) are typically symmetric around $l = 0$. Truncating the impulse response to $-L/2 < k < L/2$ still results in an acausal filter, so the impulse response function typically gets shifted by $L/2$, corresponding to a time delay of $L\Delta t/2$.

(More to come...)

1.2.4 Filters with infinite impulse response (IIR)

A filter with infinite impulse response that is very easy to implement is a first order low pass filter with a cutoff frequency ω_C, with a transfer function

$$\tilde{h}(\omega) = \frac{1}{1 + i\omega/\omega_C} \tag{1.33}$$

with impulse response for a continuous time filter

$$h(t) = \begin{cases} e^{-\omega_C t}\,, & t > 0\,, \\ 0\,, & t \leq 0 \end{cases} \,. \tag{1.34}$$

When sampled at intervals $t = k\Delta t$, the response function becomes a geometric series,

$$h_k \propto a^k\,, \quad \text{with} \quad a = e^{-\omega_C \Delta t} = e^{-\omega_C \frac{2\pi}{\omega_S}} \quad \text{for} \quad k > 0\,. \tag{1.35}$$

Such a response can be easily implemented by subtracting a fraction of the content of a register from itself every time step Δt, with an iteration rule $h_k = a \cdot h_{k-1}$. The iteration rule can be easily expanded to add a sampled signal f_t, leading to a recursion relation implementing the complete filter:

$$g_k = a \cdot g_{k-1} + f_k\,, \quad \text{with} \quad 0 < a < 1 \tag{1.36}$$

For small cutoff frequencies $\omega_C \ll \omega_S$, $a = 1 - \epsilon$ is close to 1, with $\epsilon \approx 2\pi\omega_C/\omega_S$. This recursion relation can be properly normalized to a unit stationary gain:

$$g_k = a \cdot g_{k-1} + \frac{1}{1-a} f_k = (1-\epsilon) g_{k-1} + \epsilon f_k\,. \tag{1.37}$$

A formal implementation, and its Fourier domain signal flow, is shown in figure 1.10.

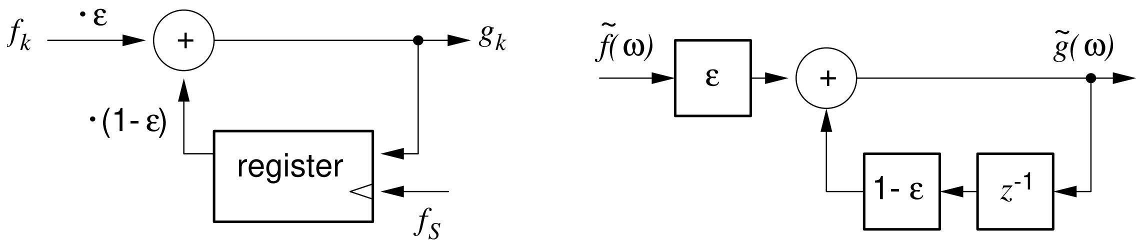

Figure 1.10: Symbolic implementation of a first order low pass filter as an example of an infinite impulse response (IIR) filter, with its associated symbolic signal flow scheme in the Fourier domain.

Such a filter is particularly efficient to implement for coefficients $\epsilon = 2^{-k}$, as multiplication by such a coefficient can be implemented by a simple bit shift (see figure 1.11). The implementation requires only bit shifts (a cheap resource), two additions, and a register. The normalizing factor in figure 1.11 has been moved to the output, but can be omitted when working with integers. As the output is extracted after the storage element, it also adds an additional delay by one sampling time. The nominal cutoff frequency is

$$\omega_C = -\frac{\omega_s}{2\pi} \ln(1 - 2^{-k}) \approx \frac{2^{-k}}{2\pi} \omega_S \,. \tag{1.38}$$

The exact filter gain $\tilde{h}(\omega)$ for such a time-discrete filter can be obtained from the recursion equation in the Fourier domain from figure 1.10:

$$\tilde{g} = (1 - \epsilon) \cdot z^{-1} \tilde{g} + \epsilon \tilde{f} \,, \quad \text{or} \quad \tilde{h}(\omega) = \frac{\epsilon}{1 - (1 - \epsilon) \cdot z^{-1}} \quad \text{with} \quad z^{-1} = e^{-i\omega\Delta t} \tag{1.39}$$

The filter gain and phase for a few values of k is shown in a Bode plot figure 1.12. While for the lower frequency parts, the filter gain is very similar to a corresponding continuous time first order low pass filter, with a unity gain at low frequencies, and a slope in the stop band of 6 dB/octave. However, there are a few notable differences:

- The gain magnitude is minimal at half of the sampling frequency, and increases again in the second Nyquist zone. This is reflecting the periodicity in the filter gain $\tilde{h}(\omega)$.

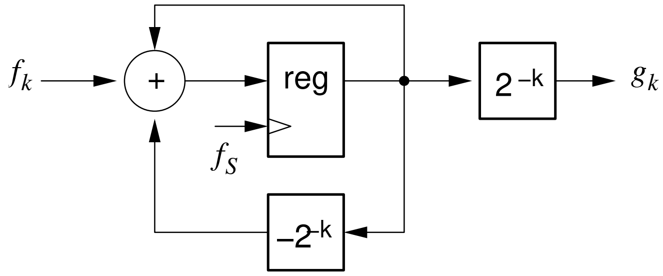

Figure 1.11: Implementation of a first order low pass filter with coefficients $\epsilon = 2^{-k}$.

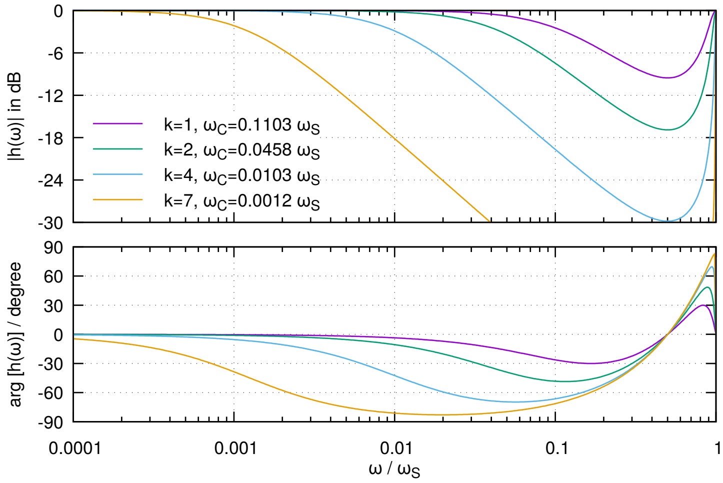

Figure 1.12: Bode diagram of the filter gain $\tilde{h}(\omega)$ of the first order low pass filter with feedback coefficients $\epsilon = 2^{-k}$.

If the gain is plotted linear in frequency, it appears mirrored at half the sampling frequency.

- The phase lag of the filtered signal behaves significantly different from its continuous-time counterpart: the phase shift becomes exactly zero at half of the sampling frequency, and then reverses sign in the second Nyquist zone. This is because the second Nyquist zone reflects the negative frequency part of the first zone, where the phase shift changes sign. And unless the cutoff frequency is significantly smaller than the sampling frequency, it does not reach the asymptotic high frequency limit of -90°.

(More to come on higher order filters)

1.3 Time-discrete mixers

In time-continuous signal processing, the frequency spectrum of a signal $f(t)$ can be shifted by multiplying a signal with a "local" harmonic oscillation at a frequency ω_L:

$$g(t) = f(t) \cdot m(t) \quad \text{with} \quad m(t) = \cos(\omega_L t) \tag{1.40}$$

The effect of this process on the spectrum can be understood by directly calculating the spectrum of the product using the convolution theorem:

$$\tilde{g}(\omega) = \mathcal{F}\left[g(t)\right] = \mathcal{F}\left[f(t) \cdot m(t)\right] = \frac{1}{2\pi}\left[\tilde{f}(\omega) * \tilde{m}(\omega)\right] . \tag{1.41}$$

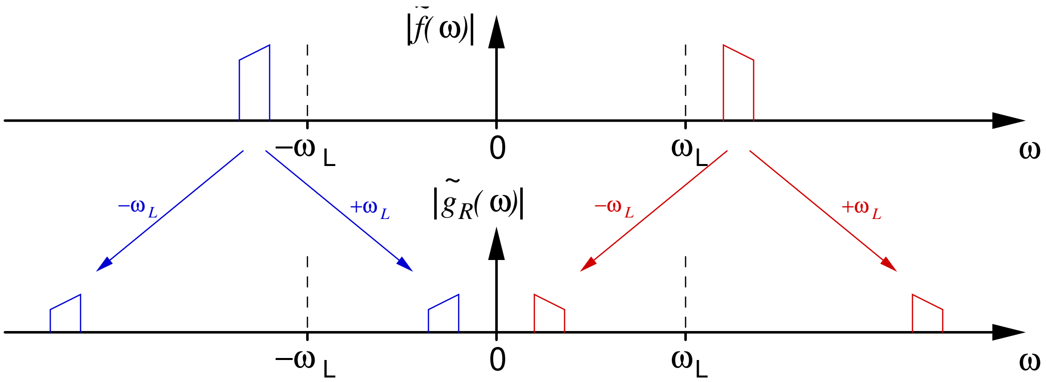

Figure 1.13: Transformation of a spectrum $\tilde{f}(\omega)$ through multiplication of $f(t)$ with a local oscillator $m(t)$ at frequency ω_L.

With the spectrum of the "local oscillator",

$$\tilde{m}(\omega) = \mathcal{F}\left[\frac{1}{2}(e^{i\omega_L t} + e^{-i\omega_L t})\right] = \pi\left(\delta(\omega - \omega_L) + \delta(\omega + \omega_L)\right) , \tag{1.42}$$

this results in a simple expression for the spectrum of the product,

$$\begin{aligned} \tilde{g}(\omega) &= \frac{1}{2\pi}\left[\tilde{f}(\omega) * \pi\left(\delta(\omega - \omega_L) + \delta(\omega + \omega_L)\right)\right] \\ &= \frac{1}{2}\left(\tilde{f}(\omega - \omega_L) + \tilde{f}(\omega + \omega_L)\right) , \end{aligned} \tag{1.43}$$

which is just a sum of two copies of the original spectrum, displaced by the local oscillator frequency ω_L in each direction. For signal of limited spectral distribution, this transformation is visualized in Figure 1.13.

For a time-discrete signal f_k, the mechanism works in exactly the same way, as the derivation of the spectrum $\tilde{g}(\omega)$ just uses the convolution theorem. However, the spectrum of a time-discrete signal contains periodic copies of the spectral component of interest, with a period of the sampling frequency ω_S. The spectrum of the product and the local oscillator contains more components that need to be taken care of.

For the simplest case, consider a local oscillator frequency that is a quarter of the sampling frequency, $\omega_L = \omega_S/4$. This has the advantage that the time-domain signal of the local oscillator has a very simple form. Possible choices (shown in figure 1.14) are:

$$m_k = \begin{cases} +1 & \text{if} \quad k \mod 4 = 0 \\ -1 & \text{if} \quad k \mod 4 = 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{or} \qquad m'_k = \begin{cases} +1 & \text{if} \quad k \mod 4 = 0,1 \\ -1 & \text{if} \quad k \mod 4 = 2,3 \end{cases} \tag{1.44}$$

This first signal corresponds to a sampled version of a cosine function,

$$m(t) = \cos(\omega_L t) \cdot Ш_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} m_k \delta(t - k \cdot \Delta t) , \tag{1.45}$$

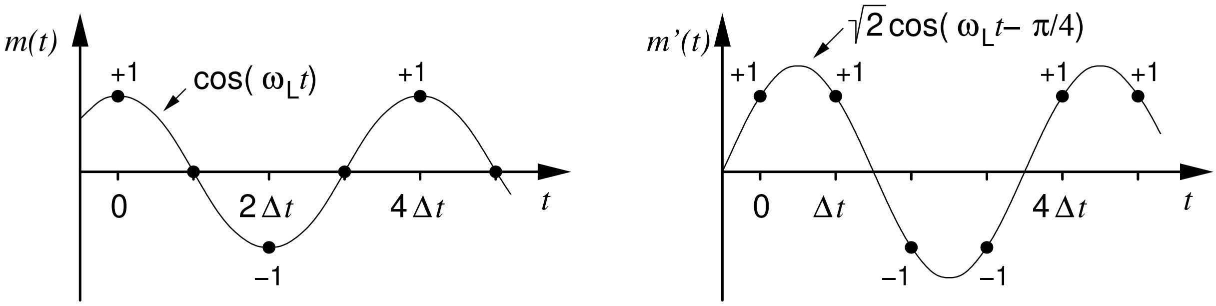

Figure 1.14: Sampled local oscillators $m(t)$ and $m'(t)$ at frequency $\omega_L = \omega_S/4$.

the second one to a sampled cosine function with a phase shift:

$$m'(t) = \sqrt{2}\cos(\omega_L t - \frac{\pi}{4}) \cdot Ш_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} m'_k \delta(t - k \cdot \Delta t) \tag{1.46}$$

The corresponding spectrum of the local oscillator can directly be obtained from (1.45):

$$\begin{aligned} \tilde{m}(\omega) &= \mathcal{F}\left[\cos(\omega_L t) \cdot Ш_{\Delta t}(t)\right] \\ &= \frac{1}{2\pi}\left(\pi(\delta(\omega - \omega_L) + \delta(\omega + \omega_L))\right) * \left(\omega_S \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S)\right) \\ &= \omega_S \sum_{k=-\infty}^{\infty} \frac{1}{2}\left(\delta(\omega - k\omega_S + \omega_L) + \delta(\omega - k\omega_S - \omega_L)\right), \end{aligned} \tag{1.47}$$

which is a Dirac comb in frequency with peaks at odd multiples of ω_L. The expression for the resulting spectrum is again obtained by convolution of the local oscillator spectrum with the signal spectrum:

$$\begin{aligned} \tilde{g}(\omega) &= \tilde{f}(\omega) * \tilde{m}(\omega) = \int_{-\infty}^{\infty} \tilde{f}(\omega - \omega') \cdot \tilde{m}(\omega') d\omega' \\ &= \int_{-\infty}^{\infty} \tilde{f}(\omega - \omega') \cdot \left[\omega_S \sum_{k=-\infty}^{\infty} \frac{1}{2}\left(\delta(\omega' - k\omega_S + \omega_L) + \delta(\omega' - k\omega_S - \omega_L)\right)\right] d\omega' \\ &= \omega_S \sum_{k=-\infty}^{\infty} \frac{1}{2}\left(\tilde{f}(\omega + \omega_L - k\omega_S) + \tilde{f}(\omega - \omega_L - k\omega_S)\right) \end{aligned} \tag{1.48}$$

The multiplication of the time-discrete signal f_k with the local oscillator m_k then generates copies of the original spectrum, displaced by $\pm\omega_L$. Assuming that the original signal has its spectral content in a small band around the local oscillator frequency, that band will be moved to frequency 0 (baseband), and another copy of it around $2\omega_L = \omega_S/2$, just at the border between Nyquist zones 1 and 2 (see figure 1.15). The spectrum

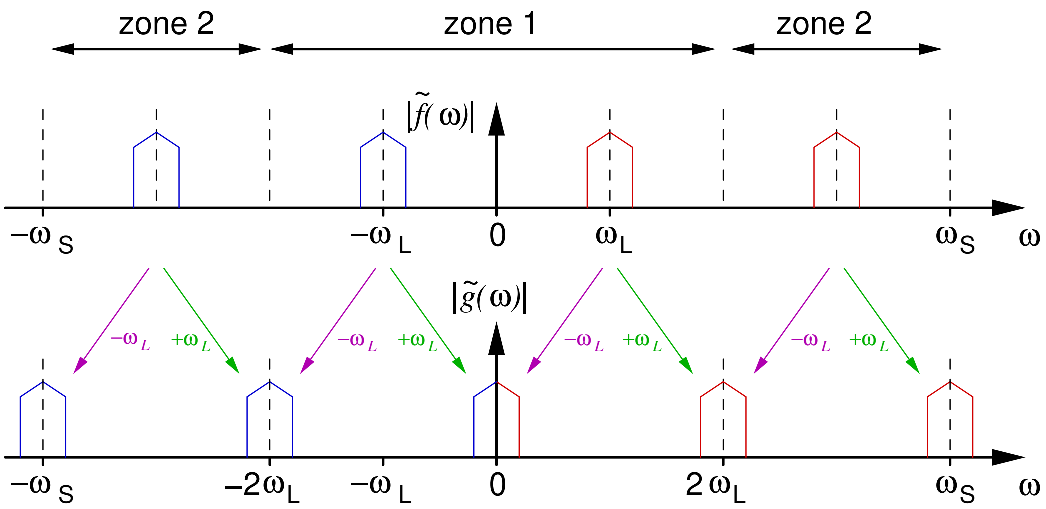

Figure 1.15: Transformation of a spectrum $\tilde{f}(\omega)$ by multiplying with a time-discrete local oscillator at a quarter of the sampling frequency ω_L.

$\tilde{g}(\omega)$ is again a series of small spectral bands as $\tilde{f}(\omega)$, but this time appearing at even multiples of ω_L. This looks like a spectrum that could be sampled at half the sampling frequency, without loosing any information contained in the product spectrum, as long as the interesting bandwidth in the original spectrum is narrower than ω_L. Such a process is called *decimation*. If the original sampling rate is to be retained, the contributions near $\omega_S/2$ have to be suppressed by proper filtering.

The original signal spectrum in a practical system $\tilde{f}(\omega)$ will also contain components near $\omega = 0$, be it a simple constant offset, or very slowly varying contributions to the sampled signal. These undesired spectral components will be moved to regions around $\omega_L + k\omega_S$, and $-\omega_L + k\omega_S$ for all integer $k \in \mathbb{Z}$, populating regions at odd multiples of ω_L. Therefore, appropriate filtering techniques have to be applied in order to suppress these contributions.

So far, a mixing scenario was considered with a local oscillator frequency $\omega_L = \omega_S/4$. This results in the largest bandwidth that can be transferred to the baseband, and is particularly easy to implement numerically, as the multiplication process is restricted to values of $m_k = \pm 1$, which just requires a sign change. However, if other frequencies are desired, other sinusoidal local oscillator values m_k can be used in a similar way. Alternatively, other local oscillator waveforms, e.g. rectangular signals with different frequencies with values $m_k \pm 1$ can be used. Such approaches can be useful to implement lock-in amplifiers or receiver designs where the mixing process is carried out numerically at a fixed sampling rate in a signal processor for different target frequencies. The derivation of the resulting spectral shift can be carried out in the same way, with a local oscillator spectrum $\tilde{m}(\omega)$ that may contain higher harmonics of ω_L.

1.4 Decimation of time-discrete signals

If the information of a time-discrete signal f_k is contained in a sufficiently small spectral band in $\tilde{f}(\omega)$, there is a redundancy in the samples. This suggests that only a subset of the samples need to be taken. In the following, we consider to a situation where the sampling rate is reduced by a factor of M by keeping only every M-th sample, and skipping the $M-1$ samples in between. The decimated time-discrete signal f'_k will have a new sampling frequency $\omega'_S = \omega_S/M$, resulting in a more densely spaced spectrum, with a lower periodicity:

$$\tilde{f}'(\omega) = \tilde{f}'(\omega + k \cdot \omega_S/M)\,, \quad k \in \mathbb{Z} \tag{1.49}$$

The sampling can be understood by introducing a decimating signal d_k:

$$f'_k = f_k \cdot d_k\,, \quad \text{with} \quad d_k = \begin{cases} 1 & \text{for} \quad k \mod M = 0\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.50}$$

This decimating signal d_k can be expressed through its discrete Fourier transformation,

$$d_k = \frac{1}{M} \sum_{l=0}^{M} e^{2\pi i \frac{l}{M}}\,, \tag{1.51}$$

which can be used to evaluate the spectrum of the decimated signal:

$$\begin{aligned} \tilde{f}'_R(\omega) &= \sum_{k=-\infty}^{\infty} f'_k e^{-i\omega\Delta tk} = \sum_{k=-\infty}^{\infty} f_k d_k e^{-i\omega\Delta tk} \\ &= \sum_{k=-\infty}^{\infty} f_k \frac{1}{M} \sum_{l=0}^{M} e^{2\pi i \frac{l}{M}} e^{-i\omega\Delta tk} \\ &= \frac{1}{M} \sum_{l=0}^{M} \sum_{k=-\infty}^{\infty} f_k e^{-i(\omega - \frac{\omega_S}{M} l)\Delta tk} \\ &= \frac{1}{M} \sum_{l=0}^{M} \tilde{f}_R(\omega - \frac{\omega_S}{M} l) \end{aligned} \tag{1.52}$$

The reconstructed spectrum of the decimated signal f'_k is M copies of the spectrum $\tilde{f}_R(\omega)$ of the original signal, displaced by the multiples of the new sampling frequency ω_S/M. As the size of the Nyquist zones are reduced, an issue arises when some spectral content of the original stream overlaps with the new Nyquist zones corresponding to the decimated sampling frequency, leading to undesired artifacts in the decimated stream. Therefore, the spectral content of the original signal in these areas needs to be reduced with an adequate low pass filter to acceptable levels, in the same way as sampling time-continuous signals in the first place.

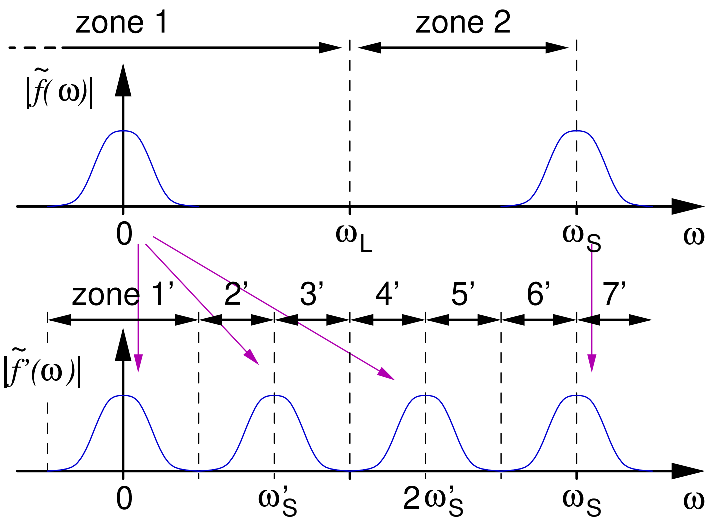

Figure 1.16: Decimating a time-discrete signal generates a set of copies of the original spectrum $\tilde{f}(\omega)$. Shown here the spectral change for decimating by $M = 3$, i.e., $\omega'_S = \omega_S/3$.

1.5 Noise sources in time-discrete systems

Noise on a signal is refered to a part of the signal that is not reproducible (or correlated with any useful signal part). A noisy signal is treated as the sum of a clean signal $f(t)$, and a stochastical noise function $n(t)$:

$$f_N(t) = f(t) + n(t) \tag{1.53}$$

The stochastic nature of the noise means requires that an ensemble average has to be carried out for the noise part. Noise is then characterized by statistical properties, such as a standard deviation, and a mean. Typically,

$$\langle n(t) \rangle = 0\,, \quad \langle n^2(t) \rangle =: \sigma_n^2 \neq 0\,. \tag{1.54}$$

Here, $\langle\cdot\rangle$ indicates an ensemble average over different noise possibilities, and σ_n is the standard deviation of the noise amplitude.

For a mathematical description, consider first the definition of a power P contained in a continuous-time signal $f(t)$, or discrete time sampled f_k,

$$P = \frac{1}{T}\int\limits_0^T f^2(t)dt \quad \text{and} \quad P = \frac{1}{N}\sum_{k=0}^{N-1} f_k^2 \tag{1.55}$$

for some averaging time intervall T or sample number N. This corresponds to the "root mean square" of the signal function $f(t)$ or time-discrete set f_k. For a noise signal $n(t)$, this quantity can also be evaluated:

$$P = \frac{1}{T}\int\limits_0^T \left\langle n^2(t) \right\rangle dt = \sigma_n^2 \tag{1.56}$$

Appendix A

Fourier transformation convention

Fourier transformations come with a variety of normalizations. This appendix lines out the definitions/conventions used in this text.

The Fourier transformation $\mathcal{F}$

$$\tilde{a}(\omega) = \mathcal{F}[a(t)] := \int_{-\infty}^{\infty} a(t) e^{-i\omega t} dt \tag{A.1}$$

converts a time-domain function $a(t)$ into a frequency domain function $\tilde{a}(\omega)$. The tilde above the function symbol is used to indicate a function that depends on a frequency (or angular frequency ω).

The inverse Fourier transformation $\mathcal{F}^{-1}$is defined in the following way:

$$a(t) = \mathcal{F}^{-1}[\tilde{a}(\omega)] := \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{a}(\omega) e^{i\omega t} d\omega \tag{A.2}$$

With this combination, the forward and backward transformation reverse each other:

$$\mathcal{F}^{-1}\left[\mathcal{F}[a(t)]\right] = a(t) \quad \text{and} \quad \mathcal{F}\left[\mathcal{F}^{-1}[\tilde{a}(\omega)]\right] = \tilde{a}(\omega) \tag{A.3}$$

A common convention used in signal processing is to replace the integration over angular frequencies $\omega = 2\pi f$ by integration over frequencies f. Then, the Fourier transformation and its inverse look more symmetric:

$$\tilde{a}(f) = \mathcal{F}[a(t)] \quad = \quad \int_{-\infty}^{\infty} a(t) e^{-2\pi i f t} dt \tag{A.4}$$

$$a(t) = \mathcal{F}^{-1}[\tilde{a}(f)] \quad = \quad \int_{-\infty}^{\infty} \tilde{a}(f) e^{2\pi i f t} df \tag{A.5}$$

However, the values $\tilde{a}(\omega)$ and $\tilde{a}(f = \omega/2\pi)$ are the same.

A.1 Fourier-transformations of real-valued signals

Most signals are real-valued functions of time, following the identity

$$f(t) = f^*(t)\,, \tag{A.6}$$

where the asterisk $*$ denotes the complex conjugate. This leads to a simple relation for the Fourier transformation:

$$f(-\omega) = \int\limits_{-\infty}^{\infty} e^{+i\omega t} f(t)dt = \left[\int\limits_{-\infty}^{\infty} e^{-i\omega t} f^*(t)dt\right]^* = \tilde{f}^*(\omega)\,, \tag{A.7}$$

This means that the Fourier transformation of real-valued functions have always positive and negative frequency components of the same magnitude.

A.2 Fourier-transformation of time-discrete signals

To obtain the Fourier transformation of a time-discrete signal, the conventional definition (A.1) can be used, replacing the continuous time-domain function by a weighted Dirac comb

$$a(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - k\Delta t)\,. \tag{A.8}$$

Then, the integration can be carried out:

$$\begin{aligned}
\tilde{a}(\omega) &= \mathcal{F}[a(t)] = \int\limits_{-\infty}^{+\infty} a(t) e^{-i\omega t}\,dt \\
&= \int\limits_{-\infty}^{+\infty} \sum_{k=-\infty}^{\infty} a_k \delta(t - k\Delta t)\, e^{-i\omega t}\,dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int\limits_{-\infty}^{+\infty} e^{-i\omega t} \delta(t - k\Delta t)dt\,, \quad t' = t - k\Delta t \\
&= \sum_{k=-\infty}^{\infty} a_k\, e^{-i\omega k\Delta t} \int\limits_{-\infty}^{+\infty} e^{-i\omega t'} \delta(t')dt' \\
&= \sum_{k=-\infty}^{\infty} a_k\, e^{-i\omega\Delta t\cdot k}.
\end{aligned} \tag{A.9}$$

The Fourier transformation is periodic with a period of the sampling frequency ω_S:

$$\tilde{a}(\omega) = \tilde{a}(\omega + k\cdot 2\pi/\Delta t) = \tilde{a}(\omega + k\cdot\omega_S) \quad \text{for} \quad k \in \mathbb{Z}\,. \tag{A.10}$$

The inverse Fourier transformation for time-discrete signals makes use of the periodicity in the Fourier domain:

$$
\begin{aligned}
a(t) &= \mathcal{F}^{-1}[\tilde{a}(\omega)] = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \tilde{a}(\omega)e^{i\omega t}\,d\omega \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int\limits_{-\omega_S/2}^{\omega_S/2} \tilde{a}(\omega)e^{i(\omega+k\omega_S)t}d\omega \\
&= \frac{1}{2\pi}\int\limits_{-\omega_S/2}^{\omega_S/2} \tilde{a}(\omega)\left[\sum_{k=-\infty}^{\infty} e^{ik\omega_S t}\right]e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int\limits_{-\omega_S/2}^{\omega_S/2} \tilde{a}(\omega)e^{i\omega t}\left[\sum_{k=-\infty}^{\infty} \delta(t-k\Delta t))\right]d\omega \\
&= \sum_{k=-\infty}^{\infty} \delta(t-k\Delta t))\underbrace{\frac{1}{2\pi}\int\limits_{-\omega_S/2}^{\omega_S/2} \tilde{a}(\omega)e^{i\omega k\Delta t}d\omega}_{=:a_k}
\end{aligned}
\tag{A.11}
$$

The last line makes use of a representation of the sampled function, and can be read as a definition of the time-discrete inverse Fourier transform:

$$
a_k = \frac{1}{2\pi}\int\limits_{-\omega_S/2}^{\omega_S/2} \tilde{a}(\omega)e^{i\omega k\Delta t}d\omega \quad \text{with} \quad k \in \mathbb{Z}\,. \tag{A.12}
$$

A.3 Parseval's theorem, power and total energy of signals

For square-integrable functions $f(t), g(t)$, i.e., functions that do not have an infinite extent, Parseval's theorem states

$$
\int\limits_{-\infty}^{\infty} |f(t)\cdot g^*(t)|\,dt = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \tilde{f}(\omega)\cdot\tilde{g}^*(\omega)d\omega\,, \tag{A.13}
$$

This allows to define a total energy E of a signal,

$$
E = \int\limits_{-\infty}^{\infty} \left|f(t)^2\right|dt = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left|\tilde{f}^2(\omega)\right|d\omega\,. \tag{A.14}
$$

The integrand in the second integral can be interpreted as a *spectral power density.*

For time-discrete signals, Parseval's theorem reads

$$\Delta t \sum_{k=-\infty}^{\infty} f_k g_k^* = \frac{1}{2\pi} \int_{-\omega_S/2}^{\omega_S/2} \tilde{f}(\omega) \cdot \tilde{g}^*(\omega) d\omega \,. \tag{A.15}$$

One can again define a total signal energy,

$$E = \Delta t \sum_{k=-\infty}^{\infty} \left| f_k^2 \right| = \frac{1}{2\pi} \int_{-\omega_S/2}^{\omega_S/2} \left| \tilde{f}^2(\omega) \right| d\omega \,. \tag{A.16}$$

The quantity E in this expression has the ugly property that for continuous signals, it is not finite. A connection to concepts of power and spectral power density becomes then painful.

This can be addressed by introducing a time-continuous window function $\Pi_T(t)$, or its time-discrete equivalent $\Pi_{T,k}$ for a large time window T

$$\Pi_T(t) = \begin{cases} 1\,, & -T/2 < 2 < T/2 \\ 0 & \text{otherwise} \end{cases} \,, \qquad \Pi_{T,k} = \begin{cases} 1\,, & -T/2\Delta t < 2 < T/2\Delta t \\ 0 & \text{otherwise} \end{cases} \,. \tag{A.17}$$

Its Fourier tranform in time-continuous and time-discrete versions are

$$\mathcal{F}\left[\Pi_T(t)\right] = \int_{-\infty}^{\infty} \Pi_T(t) e^{-i\omega t} dt = \int_{-T/2}^{T/2} e^{-i\omega t} dt = T \cdot \operatorname{sinc}\left(\frac{\omega T}{2}\right) \tag{A.18}$$

and

$$\begin{aligned} \mathcal{F}\left[\Pi_{T,k}\right] &= \sum_{k=-\infty}^{\infty} \Pi_{T,k} e^{-i\omega \Delta t k} = \sum_{k=-T/(2\Delta t)}^{T/(2\Delta t)} e^{-i\omega \Delta t k} \\ &= e^{i\omega T/2} \sum_{k=0}^{T/\Delta t} e^{-i\omega \Delta t k} \\ &= e^{i\omega T/2} \frac{1 - e^{-i\omega T}}{1 - e^{-i\omega \Delta t}} \\ &= e^{i\omega \Delta t/2} \frac{\sin \omega T/2}{\sin \omega \Delta t/2} \end{aligned} \tag{A.19}$$

Appendix B

Dirac comb properties

The definition of a Dirac comb of a certain separation Δt in time,

$$Ш_{\Delta t}(t) := \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t)\,, \tag{B.1}$$

deserves some clarification on the notation. The definition above includes a timing scale Δt as an index of the function. Without an index, it is supposed to be one, or dimensionless:

$$Ш(x) := \sum_{k=-\infty}^{\infty} \delta(x - k)\,. \tag{B.2}$$

The relationship between the indexed and non-indexed comb function are determined by the scaling property of the Dirac function:

$$Ш_{\Delta t}(t) = \frac{1}{\Delta t} Ш\left(\frac{t}{\Delta t}\right) \tag{B.3}$$

The Dirac comb can be represented as a Fourier series,

$$Ш_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} e^{i\frac{2\pi}{\Delta t}nt}\,, \tag{B.4}$$

Result where I want to get:

$$\tilde{F}(\omega) = \mathcal{F}\left[Ш_{\Delta t}(t)\right](\omega) = \frac{2\pi}{\Delta t} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{\Delta t}) = \omega_S \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_S) \tag{B.5}$$

Appendix C

Convolution of two functions

A very useful operation is the convolution (or convolution product) of two functions. For two functions $a(t)$ and $b(t)$, the convolution function is defined as

$$c(t) = a(t) * b(t) := \int_{-\infty}^{\infty} a(t')b(t-t')dt' = \int_{-\infty}^{\infty} a(t-t')b(t')dt' \,. \tag{C.1}$$

The last identity indicates that the convolution is a commutative operation.

The *convolution theorem* makes a statement about the Fourier transformation of the convolution of two functions:

$$\mathcal{F}[a(t) * b(t)] = \tilde{a}(\omega) \cdot \tilde{b}(\omega) \tag{C.2}$$

This means that the Fourier transformation of the convolution of two functions is the direct product of the Fourier transformations of each function.

This can be shown by explicit calculation and making use of a property of the Kronecker delta:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} dt = \delta(\omega) \tag{C.3}$$

Using the convention of Fourier transformations, the convolution of two functions in the frequency domain leads to

$$\mathcal{F}^{-1}[\tilde{a}(\omega) * \tilde{b}(\omega)] = 2\pi \, a(t) \cdot b(t) \tag{C.4}$$

or equivalently

$$a(t) \cdot b(t) = \mathcal{F}^{-1} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{a}(\omega')\tilde{b}(\omega - \omega')d\omega' \right] \tag{C.5}$$

When using the convention of integration over frequencies rather angular frequencies, both relations become more symmetric again:

$$\mathcal{F}[a(t) * b(t)] = \tilde{a}(f) \cdot \tilde{b}(f) \tag{C.6}$$

$$\mathcal{F}^{-1}[\tilde{a}(f) * \tilde{b}(f)] = a(t) \cdot b(t) \tag{C.7}$$